# Fourier decoupling

Pavel Zorin-Kranich

HCM Symposium, 2021-08-23

University of Bonn

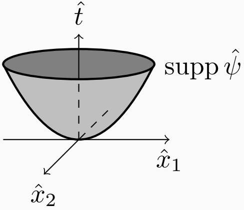
# Functions with Fourier suport on submanifolds

Free Schrödinger equation: $2\pi i \partial_t \psi = \Delta_x \psi, \qquad \psi(x,0) = g(x)$

Solution: $\mathcal{F}_x$ : $2\pi i \partial_t \mathcal{F}_x \psi = (2\pi i \xi)^2 \mathcal{F}_x \psi(\xi, t)$

ODE: $\mathcal{F}_x \psi(\xi, t) = e^{2\pi i |\xi|^2 t} \mathcal{F}_x \psi(\xi, 0)$

$\mathcal{F}_x^{-1}$ : $\psi(x,t) = \int e^{2\pi i (|\xi|^2 t + \xi x)} \hat{g}(\xi)\, \mathrm{d}\xi.$



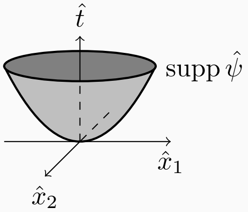
# Functions with Fourier suport on submanifolds

Free Schrödinger equation: $2\pi i \partial_t \psi = \Delta_x \psi, \qquad \psi(x,0) = g(x)$

Solution: $\mathcal{F}_x$ : $2\pi i \partial_t \mathcal{F}_x \psi = (2\pi i \xi)^2 \mathcal{F}_x \psi(\xi, t)$

ODE: $\mathcal{F}_x \psi(\xi, t) = e^{2\pi i |\xi|^2 t} \mathcal{F}_x \psi(\xi, 0)$

$\mathcal{F}_x^{-1}$ : $\psi(x,t) = \int e^{2\pi i (|\xi|^2 t + \xi x)} \hat{g}(\xi)\, \mathrm{d}\xi.$




**Theorem (Strichartz 1977)**

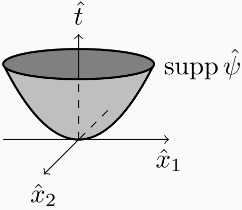
$$\|\psi\|_{L^{2+4/d}(\mathbb{R}^{d+1})} \lesssim \|g\|_{L^2(\mathbb{R}^d)}.$$

Free Schrödinger equation: $2\pi i \partial_t \psi = \Delta_x \psi, \qquad \psi(x,0) = g(x)$

Solution: $\mathcal{F}_x$ : $2\pi i \partial_t \mathcal{F}_x \psi = (2\pi i \xi)^2 \mathcal{F}_x \psi(\xi, t)$

ODE: $\mathcal{F}_x \psi(\xi, t) = e^{2\pi i |\xi|^2 t} \mathcal{F}_x \psi(\xi, 0)$

$\mathcal{F}_x^{-1}$ : $\psi(x,t) = \int e^{2\pi i (|\xi|^2 t + \xi x)} \hat{g}(\xi) \, \mathrm{d}\xi.$



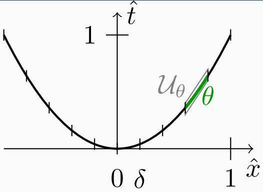

**Theorem (Strichartz 1977)**

$$\|\psi\|_{L^{2+4/d}(\mathbb{R}^{d+1})} \lesssim \|g\|_{L^2(\mathbb{R}^d)}.$$

Fourier restriction formulation:

$$\|\psi\|_{L^{2+4/d}(\mathbb{R}^{d+1})} \lesssim \|\hat{g}\|_{L^2(\mathbb{R}^d)}.$$

**Theorem (Bourgain, Demeter 2014)**
*Let $\theta$ be $\delta$-caps on the unit paraboloid.*
*Let $\mathcal{U}_\theta \supset \theta$ be $\delta \times \delta^2$-boxes.*
*Then, for any functions with* supp $\widehat{f_\theta} \subset \mathcal{U}_\theta$,




$$\Big\|\sum_\theta f_\theta\Big\|_{L^{2+4/d}(\mathbb{R}^{d+1})} \lessapprox \Big(\sum_\theta \|f_\theta\|^2_{L^{2+4/d}(\mathbb{R}^{d+1})}\Big)^{1/2}.$$

$\lessapprox$ means "$\leq C_\varepsilon \delta^{-\varepsilon}$" for every $\varepsilon > 0$.

With $L^2$ in place of $L^{2+4/d}$, this is Plancherel's theorem.

To recover (up to $\delta^{-\varepsilon}$ loss) Strichartz estimate, take

$$\widehat{f_\theta} = \int_\theta g_\theta(\xi)\widehat{\varphi}(\cdot - \xi)\,\mathrm{d}\xi,$$

$\varphi \approx \mathbf{1}_{B(0,\delta^{-2})}$ smooth. Then, with $p = 2 + 4/d$,

$$\|f_\theta\|_{L^p(\mathbb{R}^{d+1})} \leq \|\widehat{f_\theta}\|_{L^{p'}(\mathbb{R}^{d+1})} \leq \delta^{-2/p}\|g_\theta\|_{L^{p'}(\mathbb{R}^d)} \leq \|g_\theta\|_{L^2(\mathbb{R}^d)},$$

where we used Hausdorff–Young and Hölder's inequalities.

# Applications of decoupling

Decoupling for the paraboloid is like localized Strichartz estimates.

- Local smoothing for the wave equation (paraboloid $\to$ cone)
  Sogge, Seeger, Stein, Mockenhaupt 90s, Wolff, Tao, Vargas, Vega, Garrigós, Seeger 00s, Bourgain, Demeter 10s
- Strichartz estimates on manifolds
  Beltran, Hickman, Sogge
- Maximal estimates for Schrödinger equation
  Carleson, Sjölin 70s, Kenig, Ponce, Vega 90s, Guth, X. Li, X. Du, R. Zhang, H. Wang

# Applications of decoupling

Decoupling for the paraboloid is like localized Strichartz estimates.

- Local smoothing for the wave equation (paraboloid $\to$ cone)
  Sogge, Seeger, Stein, Mockenhaupt 90s, Wolff, Tao, Vargas, Vega, Garrigós, Seeger 00s, Bourgain, Demeter 10s
- Strichartz estimates on manifolds
  Beltran, Hickman, Sogge
- Maximal estimates for Schrödinger equation
  Carleson, Sjölin 70s, Kenig, Ponce, Vega 90s, Guth, X. Li, X. Du, R. Zhang, H. Wang

Decoupling inequalities for polynomial surfaces of higher degrees.

- Vinogradov mean value theorem
  Vinogradov 30s, Arkhipov, Karatsuba, Chubarikov 80s, Wooley 90s–, Bourgain, Demeter, Guth 2014, Guo, Li, Oh, Yung, Zhang, ZK

**Question**
For a tuple $\Phi$ of polynomials in $d$ variables, how large is

$$\int_{[0,1]^\Phi} \Big| \sum_{\xi_1,\dots,\xi_d=1}^{N} e\Big(\sum_{\varphi\in\Phi} \alpha_\varphi \varphi(\xi)\Big)\Big|^p \, \mathrm{d}\alpha? \tag{*}$$

Large sieve: estimates for this mean value $\implies$ pointwise estimates

**Example**
Vinogradov: $\Phi = \{\xi, \dots, \xi^k\}$
Arkhipov, Karatsuba, Chubarikov: $\Phi = \{\xi_1^{j_1} \dots \xi_d^{j_d}, j_1, \dots, j_d \le k\}$
Parsell: $\Phi = \{\xi_1^{j_1} \dots \xi_d^{j_d}, j_1 + \dots + j_d \le k\}$

**Question**
For a tuple $\Phi$ of polynomials in $d$ variables, how large is

$$\int_{[0,1]^{\Phi}} \Big| \sum_{\xi_1,\dots,\xi_d=1}^{N} e\Big(\sum_{\varphi\in\Phi} \alpha_\varphi \varphi(\xi)\Big)\Big|^p \,\mathrm{d}\alpha? \qquad (*)$$

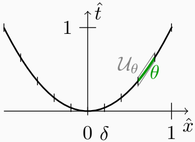
**Theorem (Parsell, Prendiville, Wooley 2012)**
*If $\Phi$ translation-dilation invariant and $p$ even integer, then*

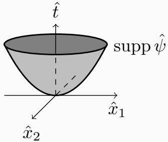
$$(*) \lessapprox N^{pd-\sum_{\varphi\in\Phi} \deg\varphi} \quad \textit{for} \quad p \geq 2|\Phi|(\max_{\varphi\in\Phi} \deg\varphi + 1).$$

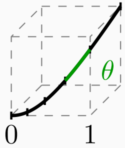
This exponent of $N$ is minimal (rectangular box around $\alpha = 0$).

In the Vinogradov case, this gives $p \geq 2k(k+1)$.

# Decoupling formulation



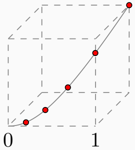




$\theta \in \mathcal{P}(\delta)$ – $\delta$-box in $\hat{x}$.
$\mathcal{U}_\theta \supseteq \theta$ box in $\hat{x}, \hat{t}$.
$\operatorname{supp} \widehat{f_\theta} \subseteq \mathcal{U}_\theta$.

Let $\Phi$ be a tuple of polynomials and partition $\{\Phi(\xi) \mid \xi \in [0,1]^d\}$ into $\delta$-caps $\theta$. How does the best constant in the decoupling inequality

$$\Big\|\sum_\theta f_\theta\Big\|_p \le D(\delta)\Big(\sum_\theta \|f_\theta\|_p^p\Big)^{1/p}, \quad \operatorname{supp} \widehat{f_\theta} \subset \mathcal{U}_\theta,$$

depend on $\delta$?

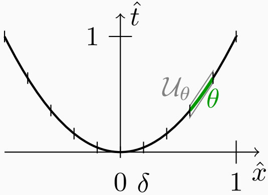


# A short history

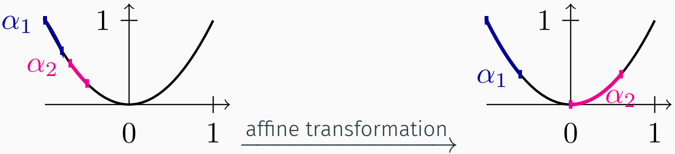
Bourgain, Demeter 2014: $\Phi = \{\xi_1^2 + \ldots + \xi_d^2\}$,
Bourgain, Demeter, Guth 2015: $\Phi = \{\xi, \ldots, \xi^k\}$,
Bourgain, Demeter, 2015: $\Phi = \{\xi, \eta, \xi^2, \xi\eta\}$,
Bourgain, Demeter, 2015: $\Phi = \{\xi^j \eta^l \mid j + l \leq 2\}$,
Bourgain, Demeter, Guo, 2016: $\Phi = \{\xi^j \eta^l \mid j + l \leq 3\}$,
Guo, Zhang 2018: $\Phi = \{\xi_1^{j_1} \cdots \xi_d^{j_d} \mid j_1 + \ldots + j_d \leq k\}$,
Guo, ZK 2018: $\Phi = \{\xi_1^{j_1} \cdots \xi_d^{j_d} \mid j_1 + \cdots + j_d \leq k, \vec{j} \leq \vec{k}\}$.
Guo, ZK 2019: $\Phi = \{\xi_1, \ldots, \xi_4, \sum_j \xi_j^2, \sum_j j\xi_j^2\}$.
Guo, Oh, Roos, Yung, ZK 2019: $\Phi = \{\xi, \eta, \zeta, \xi^2, \eta^2 + \xi\zeta\}$.
Guo, Oh, Zhang, ZK 2020: $\Phi$ quadratic.

Split this inequality ($\operatorname{supp} \widehat{f_\theta} \subseteq \mathcal{U}_\theta$):

$$\Big\|\sum_\theta f_\theta\Big\|_{L^6(\mathbb{R}^2)} \lessapprox \Big(\sum_\theta \|f_\theta\|^2_{L^6(\mathbb{R}^2)}\Big)^{1/2}$$




into these two (with $\theta \subset \alpha \subset [0,1]$):




$$\Big\|\sum_\theta f_\theta\Big\|_{L^6(\mathbb{R}^2)} \lessapprox \Big(\sum_\alpha \Big\|\sum_{\theta\subseteq\alpha} f_\theta\Big\|^2_{L^6(\mathbb{R}^2)}\Big)^{1/2}, \quad \Big\|\sum_{\theta\subset\alpha} f_\theta\Big\|_{L^6(\mathbb{R}^2)} \lessapprox \Big(\sum_{\theta\subset\alpha} \|f_\theta\|^2_{L^6(\mathbb{R}^2)}\Big)^{1/2}$$

# Bilinear reduction: Whitney decomposition

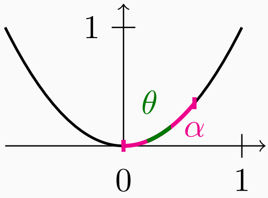
Notation for arcs $\alpha$ of length $\geq \delta$:

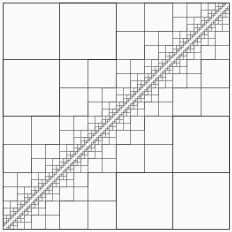
$$f_\alpha := \sum_{\theta \subset \alpha} f_\theta.$$

Whitney decomposition:

$$\Big(\sum_\theta f_\theta\Big)^2 = \sum_\theta f_\theta^2 + \sum_{\substack{\alpha_1,\alpha_2: \\ \mathrm{dist}(\alpha_1,\alpha_2) \approx |\alpha_1| = |\alpha_2|}} f_{\alpha_1} f_{\alpha_2}.$$

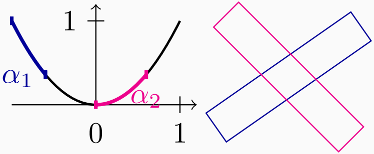
Diagonal term: easy.



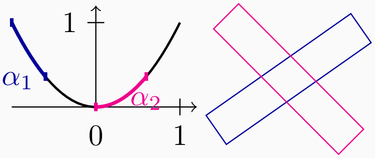
# Transversality

Parabola: transverse = separated. $\operatorname{supp}\widehat{f} \subseteq \mathcal{U}_{\alpha_1}$

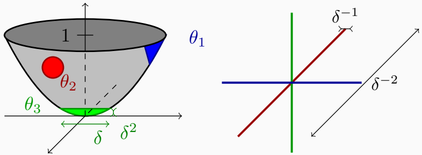


# Transversality

Parabola: transverse = separated. $\operatorname{supp} \widehat{f} \subseteq \mathcal{U}_{\alpha_1}$

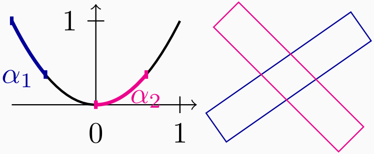



Paraboloid: transverse = not near a hyperplane.

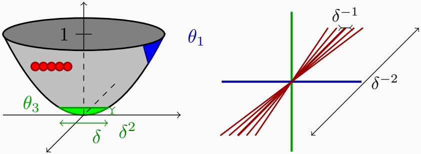



Loomis-Whitney inequality

Parabola: transverse = separated. supp $\widehat{f} \subseteq \mathcal{U}_{\alpha_1}$




Paraboloid: transverse = not near a hyperplane.

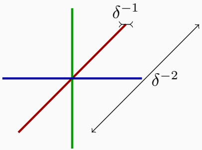



Multilinear Kakeya inequality

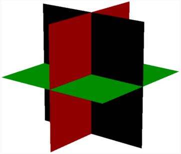
For $\pi_j : \mathbb{R}^n \to \mathbb{R}^m$, when does

$$\int_{\mathbb{R}^n} \prod_{j=1}^{M} f_j(\pi_j(x))^{\frac{n}{mM}} \, \mathrm{d}x \lesssim \prod_{j=1}^{M} \Big( \int f_j \Big)^{\frac{n}{mM}} \qquad \text{(BL)}$$

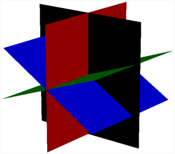
hold for all positive functions $f_j$? Picture of $\ker \pi_j$:



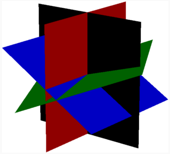

Loomis-Whitney



Fubini



bad



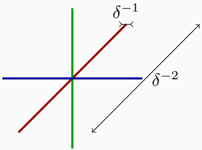
good

# Transversality: Brascamp-Lieb inequalities

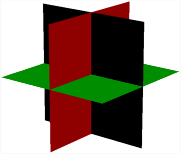
For $\pi_j : \mathbb{R}^n \to \mathbb{R}^m$, when does

$$\int_{\mathbb{R}^n} \prod_{j=1}^{M} f_j(\pi_j(x))^{\frac{n}{mM}} \,\mathrm{d}x \lesssim \prod_{j=1}^{M} \Big( \int f_j \Big)^{\frac{n}{mM}} \tag{BL}$$

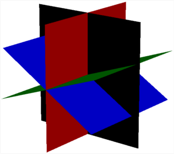
hold for all positive functions $f_j$? Picture of $\ker \pi_j$:



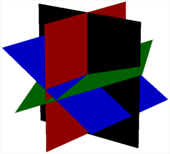

Loomis-Whitney



Fubini



bad



good

Bennett, Christ, Carbery, Tao 2008: BL inequality holds iff

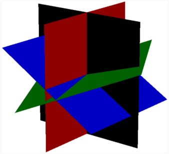
$$\dim(V) \leq \frac{n}{mM} \sum_{j=1}^{M} \dim(\pi_j V). \tag{BCCT}$$

for every subspace $V \leq \mathbb{R}^n$.

# Obtaining transversality: subvariety Bourgain-Guth

Dichotomy: broad vs. algebraic.

Broad: many papers listed under “history” are about verifying the BBCT dimension condition for a generic choice of tangent space projections $\pi_j$.



Algebraic: in the main contribution is concentrated near subvariety, induct on dimension (Bourgain+Demeter 2015 for monomials, Guo+ZK 2019 for polynomials).

# A non-transverse example

For any point on the surface

$$\Phi(r,s,t) = (r,t,s,r^2,s^2+rt),$$

tangent spaces satisfy

$$\mathrm{lin}\{(1,0,0,2r,t),(0,1,0,0,2s),(0,0,1,0,r)\} \perp (-2r,0,0,1,0).$$

Their normal spaces (2-dim), intersect a fixed 2-dim subspace.

This is non-generic in 5-dim, and BCCT condition fails.

# A non-transverse example

For any point on the surface

$$\Phi(r,s,t) = (r,t,s,r^2,s^2+rt),$$

tangent spaces satisfy

$$\mathrm{lin}\{(1,0,0,2r,t),(0,1,0,0,2s),(0,0,1,0,r)\} \perp (-2r,0,0,1,0).$$

Their normal spaces (2-dim), intersect a fixed 2-dim subspace.

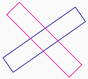
This is non-generic in 5-dim, and BCCT condition fails.

**Theorem (Guo, Oh, Roos, Yung, ZK 2019)**
*Let $\theta$ be $\delta$-caps on the surface* $\Phi(r,s,t) = (r,t,s,r^2,s^2+rt)$.

$$\Big\|\sum_\theta f_\theta\Big\|_{L^4(\mathbb{R}^5)} \lessapprox \delta^{-3/4}\Big(\sum_\theta \|f_\theta\|_{L^4(\mathbb{R}^5)}^4\Big)^{1/4}, \quad \operatorname{supp}\widehat{f_\theta} \subseteq \mathcal{U}_\theta.$$

ad-hoc proof: bilinear, two-parameter
(square caps are replaced by rectangular caps).

# Transversality: scale-dependent Brascamp-Lieb inequalities



For $\pi_j : \mathbb{R}^n \to \mathbb{R}^m$, what is the smallest $\kappa$ such that

$$\int_{B(0,R)} \prod_{j=1}^{M} f_j(\pi_j(x))^{\frac{n}{mM}} \,\mathrm{d}x \lesssim R^{\kappa} \prod_{j=1}^{M} \Big( \int f_j \Big)^{\frac{n}{mM}}$$

hold for all positive functions $f_j$ constant at scale 1?

# Transversality: scale-dependent Brascamp-Lieb inequalities

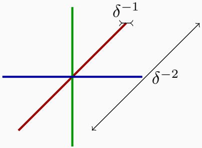
For $\pi_j : \mathbb{R}^n \to \mathbb{R}^m$, what is the smallest $\kappa$ such that

$$\int_{B(0,R)} \prod_{j=1}^{M} f_j(\pi_j(x))^{\frac{n}{mM}} \,\mathrm{d}x \lesssim R^{\kappa} \prod_{j=1}^{M} \Big(\int f_j\Big)^{\frac{n}{mM}}$$

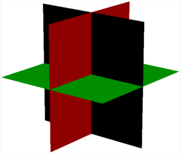
hold for all positive functions $f_j$ constant at scale 1?

Maldague 2019 (Kakeya version by ZK):

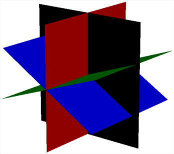
$$\kappa = \sup_{V \leq \mathbb{R}^n} \dim V - \frac{n}{mM} \sum_{j=1}^{M} \dim \pi_{V_j} V.$$



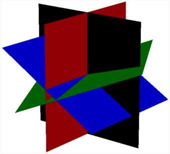

$\kappa = 0$



$\kappa = 0$



$\kappa > 0$



$\kappa = 0$

# Decoupling theorem for quadratic forms

**Theorem (Guo, Oh, Zhang, ZK 2020)**
*Let $d, n \geq 1$, and $2 \leq q \leq p < \infty$.*
*Let $\mathsf{Q} = (Q_1, \dots, Q_n)$ be quadratic forms in $d$ variables.*
*Let $\theta$ be $\delta$-caps on the manifold $S_{\mathsf{Q}} = \{(\xi, \mathsf{Q}(\xi)) : \xi \in [0,1]^d\}$. Then,*

$$\Big\|\sum_{\theta} f_\theta\Big\|_p \lessapprox \delta^{-\gamma}\Big(\sum_{\theta}\|f_\theta\|_p^q\Big)^{1/q},$$

*where*

$$\gamma = \max_{0 \leq d' \leq d} \max_{0 \leq n' \leq n} \Big(d'\big(1 - \frac{1}{p} - \frac{1}{q}\big) - \mathfrak{d}_{d',n'}(\mathsf{Q})\big(\frac{1}{2} - \frac{1}{p}\big) - \frac{2(n-n')}{p}\Big),$$

$$\mathfrak{d}_{d',n'}(\mathsf{Q}) := \inf_{\substack{M \in \mathbb{R}^{d \times d} \\ \operatorname{rank}(M) = d'}} \inf_{\substack{M' \in \mathbb{R}^{n \times n} \\ \operatorname{rank}(M') = n'}} \#_{\text{variables}}(M' \cdot (\mathsf{Q} \circ M)).$$

*The exponent $\gamma$ is the smallest possible.*

# Bilinear vs multilinear: $(r, t, s, r^2, s^2 + rt)$

$$\Big\|\sum_\theta f_\theta\Big\|_{L^4(\mathbb{R}^5)} \lessapprox \delta^{-3/4}\Big(\sum_\theta \|f_\theta\|^4_{L^4(\mathbb{R}^5)}\Big)^{1/4}.$$

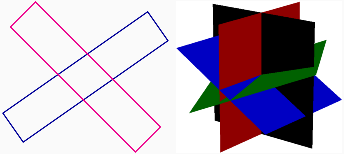
Bilinear:
two linear decouplings:
$\sigma \times 1 \times \sigma \to \sigma \times \sigma^2 \times \sigma$,
$1 \times \sigma \times 1 \to \sigma^2 \times \sigma \times \sigma^2$,
proved by bilinear methods,
applied alternatingly.

Multilinear:
multilinear ball inflation:
$\sigma \to \sigma^2$.

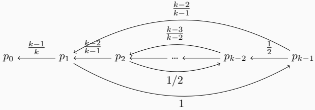
# Bilinear vs multilinear: moment curve

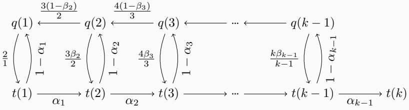
$$\Big\|\sum_{\theta} f_\theta\Big\|_{L^{k(k+1)}(\mathbb{R}^k)} \lessapprox \Big(\sum_{\theta}\|f_\theta\|^2_{L^{k(k+1)}(\mathbb{R}^k)}\Big)^{1/2}.$$

- Different ways to use same transversality (Fubini/Brascamp–Lieb):



- Different induction schemes:





- Bilinear proof is effective, because transversality is made explicit in the Vandermonde determinant.

Thanks for listening.