Second Order Arithmetic, Topological Regularity Properties, and Zermelo-Fraenkel Set Theory

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Theorem. *The following theories are equiconsistent*

- ZFC
- *Full second order arithmetic* (SOA) + *every uncountable* Π^1_1*-set of reals has a perfect subset* ($\Pi^1_1 - \mathrm{PSP}$)
- SOA + *every projective set of reals is* LEBESGUE*-measurable, has the property of* BAIRE *and, if uncountable, has a non-empty perfect subset*

These equivalences were presented at the 2003 Helsinki Logic colloquium. The first equivalence is also described in a note by DMYTRO TARANOVSKY, MIT, of 2004/5.

Ideas:

- If V is a transitive model of ZFC, then $\infty = \mathrm{Ord} \cap V$ can be viewed as an *inaccessible* cardinal. LEVY collapse ∞ to $\aleph_1$ in a class generic extension $V[G]$. In $V[G]$, every projective set of reals has strong topological regularity properties (LEBESGUE-measurability, BAIRE-measurability, perfect subset property). $V[G]$ is a model of full second order arithmetic SOA.

- Let W be a model of SOA + every uncountable Π^1_1-set of reals has a perfect subset. Using techniques of SPECKER, $\aleph_1 = \infty$ is inaccessible in L. Hence $L = L_\infty$ is a model of ZFC.

Issues:

- Class-sized Levy-collapse; can Solovay's analysis be carried out?
- Descriptive set theory in SOA.
- Building L in SOA.

Second order arithmetic SOA

- SOA formalises natural numbers as first order objects and real numbers (i.e., sets of natural numbers) as second order objects (D. Hilbert, P. Bernays).

- ”Core mathematics” can be carried out in SOA (D. Hilbert, P. Bernays).

- Reverse mathematics: fundamental theorems of mathematics are “equivalent” to *subsystems* of SOA (H. Friedman, S. Simpson, ...).

- We consider *supersystems* of SOA, but we can use techniques from Simpson.

The axioms of SOA: Basic axioms, and

- *induction*: $\forall X\,((0 \in X \wedge \forall x(x \in X \to x+1 \in X)) \to \forall x\; x \in X)$.
- *comprehension*: for every formula $\varphi(x)$ postulate

$$\exists X\,\forall x\,(x \in X \leftrightarrow \varphi(x)).$$

- *definable choice*: for every formula $\varphi(x, X)$ postulate

$$\forall x\,\exists X\,\varphi(x, X) \to \exists Z\,\forall x\,\varphi(x, (Z)_x)$$

where $(Z)_x = \{y | (x, y) \in Z\}$.

Descriptive set theory in SOA

- "Set of reals" $\mathcal{X}$ means definable, or projective set of reals.

- $\mathcal{X}$ is *countable* iff

$$\exists X \forall Y (Y \in \mathcal{X} \rightarrow \exists x\, Y = (X)_x)$$

  where $(X)_x = \{y | (x, y) \in X\}$.

- The theory of arithmetical transfinite recursion ATR_0 proves the LEBESGUE measurability of every BOREL set of reals (coded in SAO by a BOREL code) (X. YU, *Annals of Pure and Applied Logic*, 1993).

Descriptive set theory in SOA

- The perfect set theorem: "every uncountable $\mathbf{\Sigma}^1_1$-code has a non-empty perfect subset" is equivalent to arithmetical transfinite recursion ATR_0 over the base theory ACA_0 (St. Simpson, *Subsystems of Second Order Arithmetic*, Springer-Verlag, Theorem V.5.5).

- $\mathbf{\Sigma}^1_1$-bounding: if $\mathcal{C}$ is a $\mathbf{\Sigma}^1_1$-class of reals coding well-orders then the order-types of those well-orders are bounded by a countable ordinal.

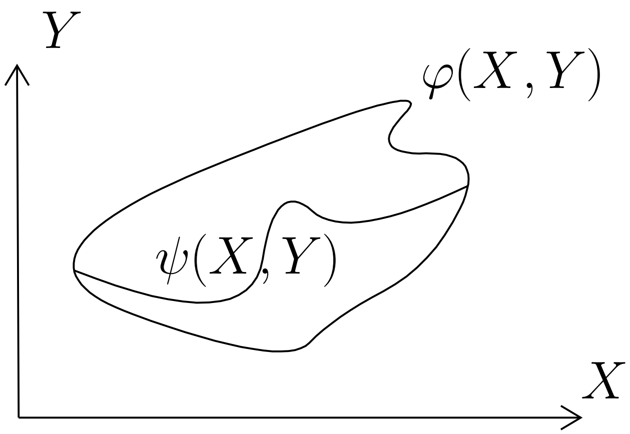
Kondo-Addison uniformization in SOA

- Simpson, VI.2.6: for every Π^1_1-formula $\varphi(X, Y)$ there is a Π^1_1-formula $\psi(X,Y)$ such that

  - $\psi(X,Y) \rightarrow \varphi(X,Y)$

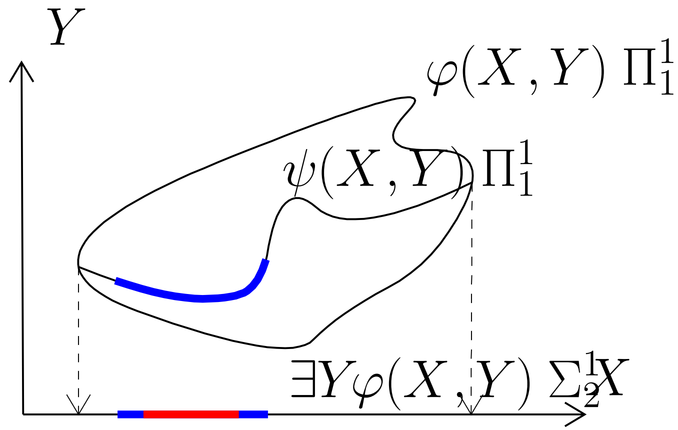
  - $\psi(X,Y) \wedge \psi(X,Y) \rightarrow Y = Y'$

  - $\exists Y \varphi(X,Y) \rightarrow \exists Y \psi(X,Y)$



Perfect subset properties (PSP) in SOA

- $\mathbf{\Pi}^1_1$-PSP implies $\mathbf{\Sigma}^1_2$-PSP



Set theory in SOA

- The theories SOA and ZFC^- + every set is countable interpret each other.

- In ZFC^-, $(\omega, \ldots, \mathbb{R})$ is a model of SOA.

- In SOA, let $\mathcal{T}$ be the set of reals which code well-founded extensional relations on ω; view such a relation as a transitive set with 0 as a distinguished element, and define an $\in$-relation E on $\mathcal{T}$. Then $(\mathcal{T}, E)$ is a model of ZFC^- + every set is countable.

- These two model constructions are canonically inverse to each other.

Constructible sets in SOA

- Inside $(\mathcal{T}, E)$ define GÖDEL's model L of constructible sets.
- $L \models \mathrm{ZFC}^-$.
- For a real x, $x \in L$ is uniformly Σ^1_2.
- For constructible reals x, y the constructible well-order $x <_L y$ of L is uniformly Δ^1_2.

$\mathbf{\Sigma}^1_2$-PSP implies the power set axiom in L

- Suffices: $\forall\alpha\exists\beta\, \mathcal{P}(\alpha)\cap L\subseteq L_\beta$

  Define $\mathcal{B}=\{X \mid X$ codes a wellorder of successor type $\wedge$ no $X'<_L X$ is isomorphic to X $\wedge$ $\exists Y\, Y$ codes the constructible level $L_{\mathrm{otp}(X)}$ $\wedge\ \mathcal{P}(\alpha)\cap L_{\mathrm{otp}(X)}\not\subseteq L_{\mathrm{otp}(X)-1}\}$

- $\mathcal{B}$ is Σ^1_2 in a code for the ordinal α.

- $\mathcal{B}$ is countable: assume not. By $\mathbf{\Sigma}^1_2$-PSP there is a perfect subset $\mathcal{C}\subseteq\mathcal{B}$. $\mathcal{C}$ is an unbounded $\mathbf{\Sigma}^1_1$ set of wellorders, contradicting the bounding theorem.

$\mathbf{Con(SOA + \Pi^1_1\text{-}PSP)}$ implies $\mathbf{Con(ZFC)}$

- $\mathrm{SOA} + \mathbf{\Pi}^1_1$-PSP implies $(\mathrm{ZFC})^L$.

LEVY collapsing the universe

- Let $(V, \in) \models \mathrm{ZFC}$, $\infty = V \cap \mathrm{Ord}$
- Extend V to $V[G]$ by class forcing with

$$\begin{aligned}\mathrm{Coll}(\infty, \aleph_1) = \{p \mid\ & p\colon \mathrm{dom}(p) \to \infty, \mathrm{dom}(p) \subseteq \infty \times \omega, \\ & p \text{ finite}, \forall (\alpha, n) \in \mathrm{dom}(p)\ p(\alpha, n) < \alpha\}\end{aligned}$$

- $\mathrm{Coll}(\infty, \aleph_1)$ is *pretame*, hence
- $V[G] \models \mathrm{ZFC}^- +$ every set is countable
- $(\omega, \mathbb{R}^{V[G]}) \models \mathrm{SOA}$

$\boldsymbol{\Pi^1_1}$-PSP in $\boldsymbol{V[G]}$

- $G \restriction (\alpha \times \omega)$ is $\mathrm{Coll}(\alpha, \aleph_1)$-generic over V.
- $V[G] = \bigcup_{\alpha < \infty} V[G \restriction (\alpha \times \omega)]$
- Let $X \in \mathbb{R}^{V[G]}$, say $X \in \mathbb{R}^{V[G \restriction (\alpha \times \omega)]}$. Then $L[X] \subseteq V[G \restriction (\alpha \times \omega)]$ and since $V[G \restriction (\alpha \times \omega)]$ satisfies the power set axiom

$$\mathbb{R} \cap L[X] \subseteq \mathbb{R} \cap V[G \restriction (\alpha \times \omega)] \in V[G]$$

- Hence in $V[G]$, there are only countably reals constructible from X. This implies Π^1_1-PSP in the parameter X.

Con(ZFC) implies Con(SOA + $\mathbf{\Pi}^1_1$-PSP)

- $V[G] \models \mathrm{ZFC}^- +$ every set is countable $+ \mathbf{\Pi}^1_1$-PSP

Shoenfield absoluteness in SOA

- Let $\varphi(X)$ be Σ^1_2. Then

$$\varphi(X) \leftrightarrow L[X] \models \varphi(X)$$

An application

- Assume $\mathrm{SOA} + \mathbf{\Pi}^1_1$-PSP
- Let $\forall X \exists Y \varphi(X, Y)$ be Π^1_4, φ Π^1_2, and $\mathrm{ZFC} \vdash \forall X \exists Y \varphi(X, Y)$
- Then $\forall X \exists Y \varphi(X, Y)$ holds (in $\mathrm{SOA} + \mathbf{\Pi}^1_1$-PSP).
  Proof. Let $X_0 \in \mathbb{R}$. $L[X_0] \models \mathrm{ZFC}$. $L[X_0] \models \forall X \exists Y \varphi(X, Y)$.
  Take $Y_0 \in L[X_0]$ such that $L[X_0] \models \varphi(X_0, Y_0)$. By absoluteness,
  $\varphi(X_0, Y_0)$. Thus $\forall X \exists Y \varphi(X, Y)$.
- Hence $\mathrm{SOA} + \mathbf{\Pi}^1_1$-PSP implies Borel determinacy.

Further results

- The LEVY collapse of the universe yields *full topological regularity*, i.e., every *projective* set of reals is LEBESGUE-measurable, has the property of BAIRE and, if uncountable, has a non-empty perfect subset