Computations on Ordinals

by Peter Koepke

University of Bonn

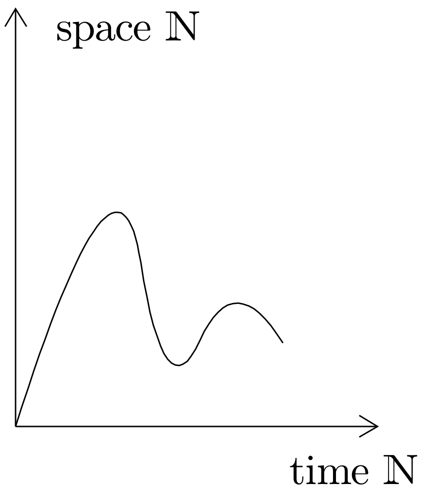
IMSC, Chennai, September 16, 2008

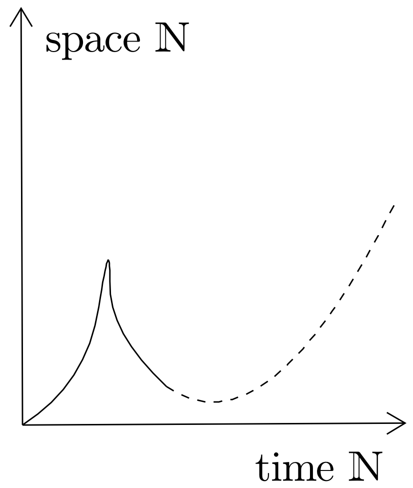
A standard TURING computation

	⋮	⋮	⋮	⋮	⋮	⋮		⋮	⋮		
	$n+1$	0	0	0	0	0		0	1	…	…
$\Uparrow$	n	0	0	0	0	0	…	1	1		
	⋮	0	0	0	0	0	…	0	0		
S	4	0	0	0	0	0	…	0	0		
P	3	0	0	0	0	0	…	0	0		
A	2	0	0	1	1	1	…	1	1		
C	1	0	1	1	0	0	…	0	0		
E	0	1	1	1	1	0	…	1	1		
		0	1	2	3	4	…	n	$n+1$	…	…

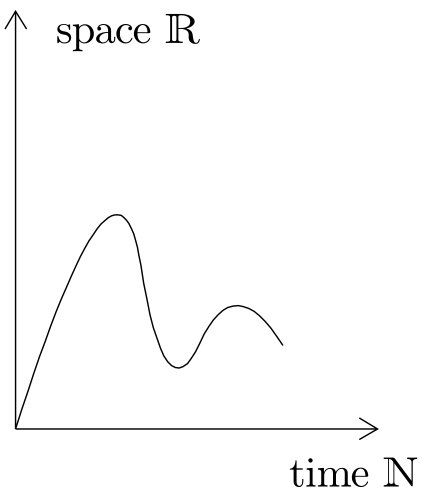
T I M E $\Rightarrow$

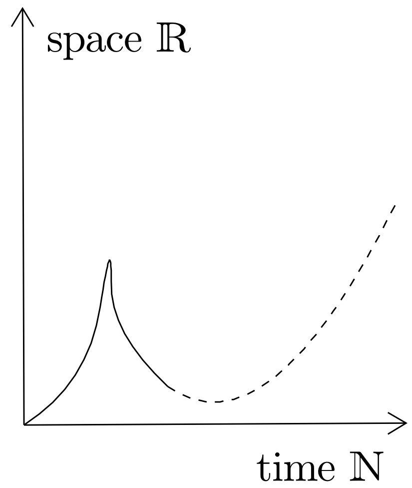
The shape of standard Turing computations



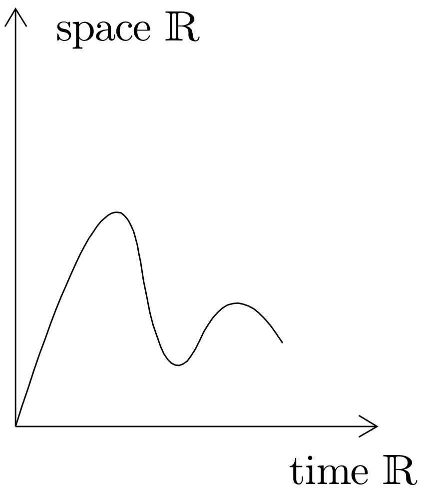


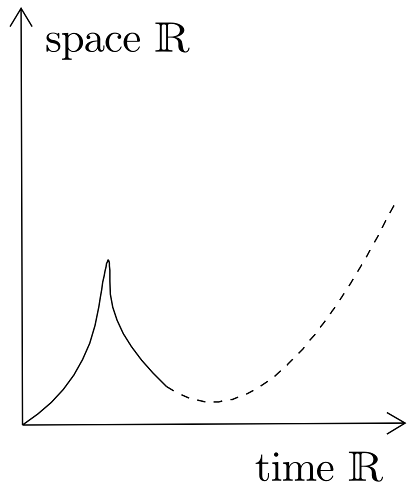
The shape of BSS computations



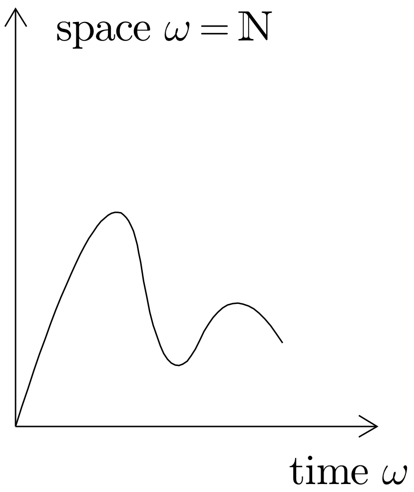


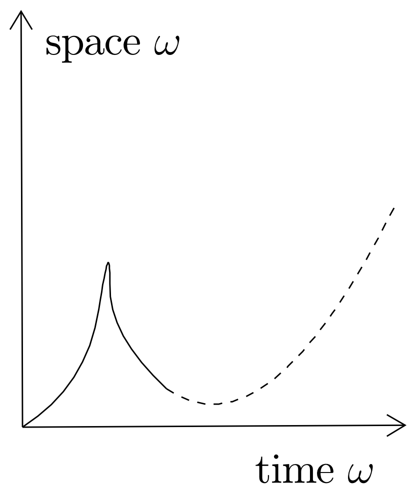
Real functions, differential equations, dynamical systems





Standard Turing computations are based on the *ordinal* $\boldsymbol{\omega} = \mathbb{N}$





Ordinals

Natural numbers:

$0 = \emptyset,\ 1 = \{0\},\ 2 = \{0, 1\},\ \ldots,\ n = \{0, 1, \ldots, n - 1\},\ \ldots$

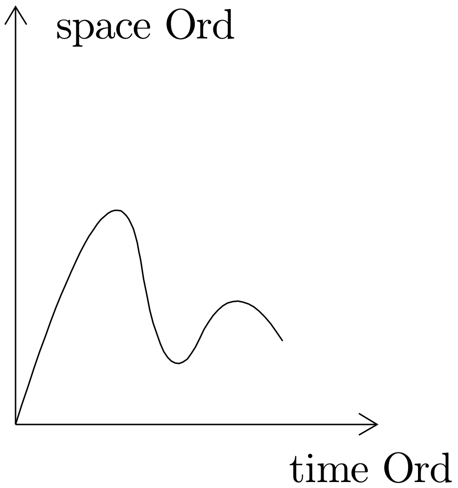
$\omega = \mathbb{N} = \{0, 1, 2, \ldots, n, \ldots\}$

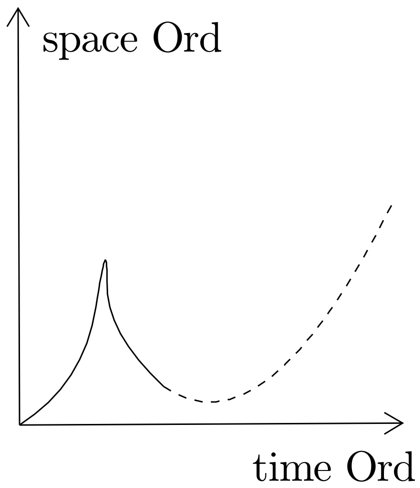
Ordinal numbers:

$0, 1, 2, \ldots, n, \ldots, \omega, \omega + 1 = \omega \cup \{\omega\}, \ldots, \alpha, \alpha + 1 = \alpha \cup \{\alpha\}, \ldots, \aleph_1, \ldots, \aleph_\omega, \ldots$

$\infty = \mathrm{Ord} = \{0, 1, 2, \ldots, \omega, \ldots, \alpha, \ldots\}$

Ordinal computations





Limit ordinals and ordinal limits

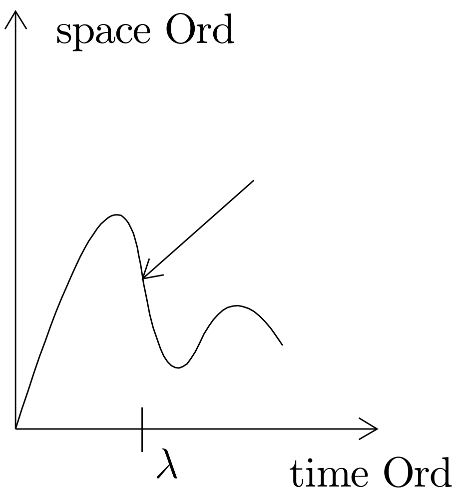
An ordinal λ is a *limit ordinal*, if it is not of the form $\lambda = 0$ or $\lambda = \mu + 1$.

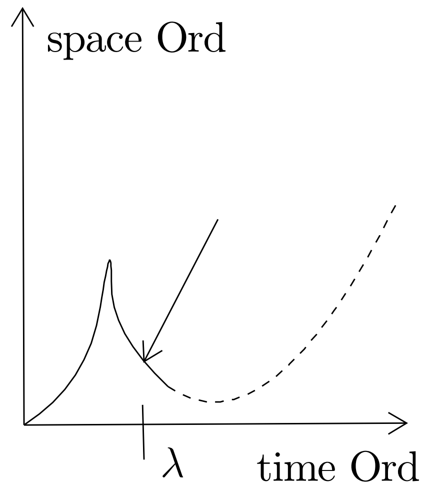
Let $\{\alpha_\xi | \xi < \lambda\} \subseteq \mathrm{Ord}$.

$\sup_{\xi<\lambda} \alpha_\xi = \bigcup_{\xi<\lambda} \alpha_\xi \in \mathrm{Ord}$, $\min_{\xi<\lambda} \alpha_\xi = \bigcap_{\xi<\lambda} \alpha_\xi \in \mathrm{Ord}$.

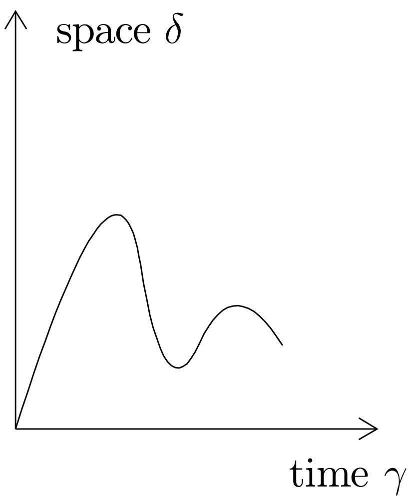
$\liminf_{\xi<\lambda} \alpha_\xi = \sup_{\zeta<\lambda} (\min_{\zeta \leqslant \xi < \lambda} \alpha_\xi)$.

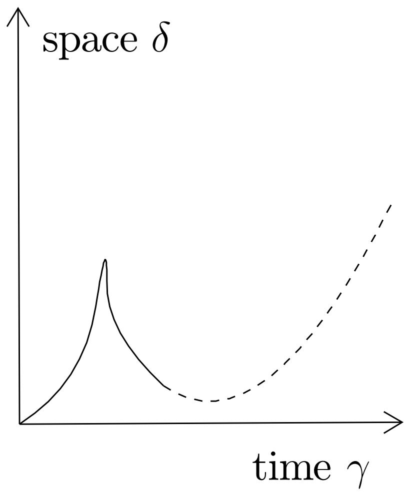
Ordinal computations: lim inf at limit ordinals



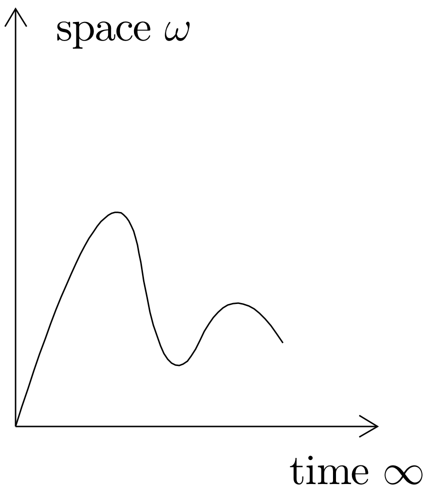


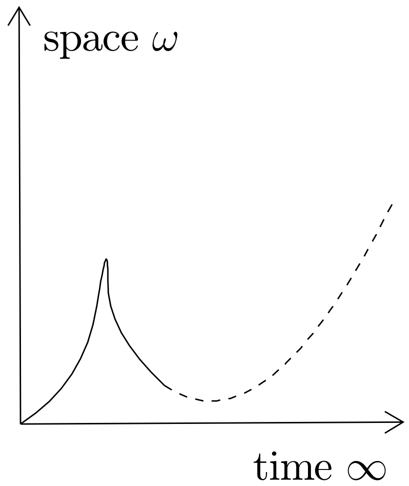
γ-δ-computations



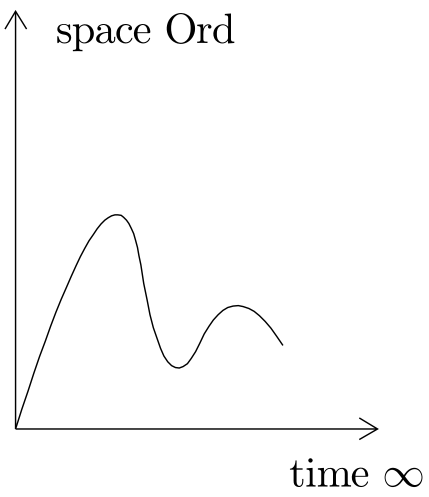


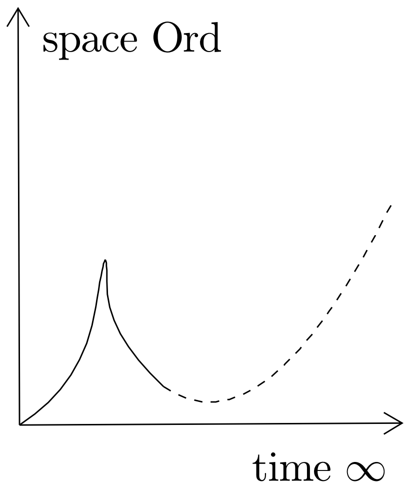
ITTM computations are ∞-ω-computations





Ordinal register machines (ORM) (with Ryan Siders)





A *register program* is a finite list $P = I_0, I_1, \ldots, I_{s-1}$ of *instructions*:

a) the *zero instruction* $Z(n)$ set register R_n to 0;

b) the *successor instruction* $S(n)$ increases register R_n by 1;

c) the *oracle instruction* $O(n)$ sets register R_n to 1 if its content is an element of the oracle, and to 0 otherwise;

d) the *transfer instruction* $T(m, n)$ sets R_n to the contents of R_m;

e) the *jump instruction* $J(m, n, q)$: if $R_m = R_n$, the register machine proceeds to the qth instruction of P; otherwise it proceeds to the next instruction in P.

Let $P = P_0, P_1, \ldots, P_{k-1}$ be a register program. A pair

$$S : \theta \to \omega, R : \theta \to ({}^{\omega}\mathrm{Ord})$$

is the ORM *computation* by P *with oracle* $Z \subseteq \mathrm{Ord}$ if:

a) θ is a successor ordinal or $\theta = \mathrm{Ord}$; θ is the *length* of the computation;

b) $S(0) = 0$; the machine starts in state 0;

c) If $t < \theta$ and $S(t) \notin s = \{0, 1, \ldots, s-1\}$ then $\theta = t + 1$; the machine *stops* if the machine state is not a program state of P;

d) If $t < \theta$ and $S(t) \in \{0, 1, \ldots, s-1\}$ then $t + 1 < \theta$; the next configuration is determined by the instruction $P_{S(t)}$: ....

e) If $t < \theta$ is a limit ordinal, the machine constellation at t is determined by taking inferior limits:

$$\begin{aligned} \forall k \in \omega \; R_k(t) &= \liminf_{r \to t} R_k(r); \\ S(t) &= \liminf_{r \to t} S(r). \end{aligned}$$

```
  ...
begin loop
    ...
    begin subloop
        ...
    end subloop
    ...
end loop
  ...
```

∞-∞-computability, Ordinal Register Machines (with Ryan Siders)

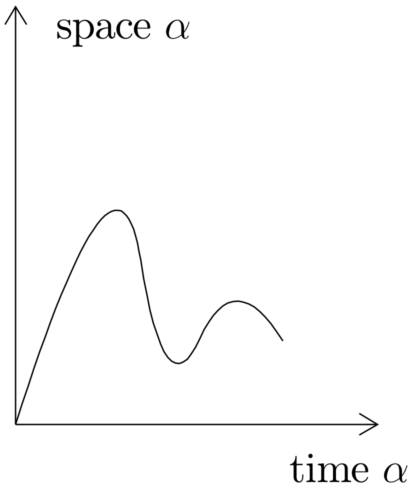
$x \subseteq \mathrm{Ord}$ is ORM *computable* (from parameters) if there are a program P and ordinals $\delta_1, \ldots, \delta_{n-1}$ such that

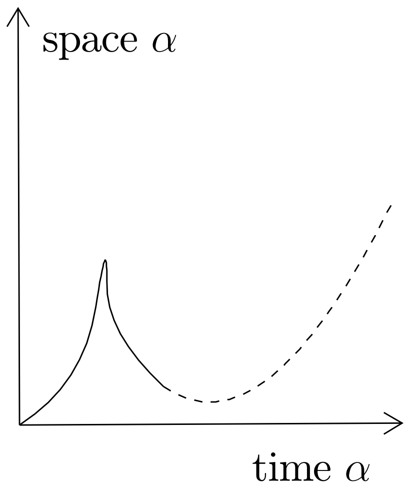
$$\forall \alpha \ P\colon (\alpha, \delta_1, \ldots, \delta_{n-1}, 0, 0, \ldots) \mapsto \chi_x(\alpha),$$

where χ_x is the characteristic function of x.

Theorem. $x \subseteq \mathrm{Ord}$ is ORM *computable* iff $x \in L$, where L is Gödel's inner model of constructible sets.

α-α-computations for admissible α (with Benjamin Seyfferth)





Theorem. Let α be an admissible ordinal and $X \subseteq \alpha$. Then

a) X is computable by an α-α-register machine in parameters $< \alpha$ iff $X \in \mathbf{\Delta}_1(L_\alpha)$

b) X is computably enumerable by an α-α-register machine in parameters $< \alpha$ iff $X \in \mathbf{\Sigma}_1(L_\alpha)$

One can characterize when a limit ordinal β is admissible using β-β-machines.
One can do parts of α recursion theory using α-α-machines, e.g., the Sacks-Simpson theorem.

Ordinal register computability

Register machines	space ω	space admissible α	space Ord
time ω	standard register machine computable $= \Delta^0_1$	-	-
time admissible α	?	α register machine (α recursion theory) computable $= \mathbf{\Delta}_1(L_\alpha)$	-
time Ord	?	?	Ordinal register machine computable $= L \cap \mathcal{P}(\mathrm{Ord})$

Ordinal TURING computability

TURING	space ω	space admissible α	space Ord
time ω	standard TURING machine computable $= \Delta^0_1$	-	-
time admissible α	?	α TURING machine (α-recursion theory) computable $= \mathbf{\Delta}_1(L_\alpha)$	-
time Ord	ITTM $\mathbf{\Delta}^1_1 \subsetneq$ computable in real parameter $\subsetneq \mathbf{\Delta}^1_2$	?	Ordinal TURING machine computable $= L \cap \mathcal{P}(\mathrm{Ord})$

Ordinal register computability

Register machines	space ω	space admissible α	space Ord
time ω	standard register machine computable $= \Delta^0_1$	-	-
time admissible α	?	α register machine (α recursion theory) computable $= \boldsymbol{\Delta}_1(L_\alpha)$	-
time Ord	ITRM Infinite time register machine computable in real parameters = ?	?	Ordinal register machine computable $= L \cap \mathcal{P}(\mathrm{Ord})$

Infinite Time Register Machines (ITRM) (with Russell Miller)

Let $P = P_0, P_1, \ldots, P_{k-1}$ be a register program. A pair

$$S\colon \theta \to \omega\,,\, R\colon \theta \to ({}^{\omega}\omega)$$

is the *infinite time register computation* by P *with oracle* $Z \subseteq \omega$ if:

a) ...

b) If $t < \theta$ is a limit ordinal, the machine constellation at t is determined by taking inferior limits **or in case of overflow resetting to** 0:

$$\begin{aligned} \forall k \in \omega\ R_k(t) &= \begin{cases} 0, \text{ if } \liminf_{r\to t} R_k(r) = \omega, \\ \liminf_{r\to t} R_k(r), \text{ else;} \end{cases} \\ S(t) &= \liminf_{r\to t} S(r). \end{aligned}$$

A subset $A \subseteq \mathcal{P}(\omega) = \mathbb{R}$ is ITRM-*computable* if there is a register program P and an oracle $Y \subseteq \omega$ such that for all $Z \subseteq \omega$:

$$Z \in A \text{ iff } P\colon (0,0,\ldots), Y \times Z \mapsto 1, \text{ and } Z \notin A \text{ iff } P\colon (0,0,\ldots), Y \times Z \mapsto 0$$

where $Y \times Z$ is the cartesian product of Y and Z with respect to the pairing function

$$(y,z) \mapsto \frac{(y+z)(y+z+1)}{2} + z.$$

Stacks

Code a stack $(r_0, \ldots, r_{m-1})$ of natural numbers by

$$r = 2^m \cdot 3^{r_0} \cdot 5^{r_1} \cdots p_m^{r_{m-1}}$$

Proposition 1. *Let $\alpha < \tau$ where τ is a limit ordinal. Assume that in some ITRM-computation using a stack, the stack contains $r = (r_0, \ldots, r_{m-1})$ for cofinally many times below τ and that all contents in the time interval (α, τ) are endextensions of $r = (r_0, \ldots, r_{m-1})$. Then at time τ the stack contents are*

$$r = (r_0, \ldots, r_{m-1}).$$

```
push 1; %% marker to make stack non-empty
      push 0; %% try 0 as first element of descending sequence
      FLAG=1; %% flag that fresh element is put on stack
Loop: Case1: if FLAG=0 and stack=0 %% inf descending seq found
          then begin; output 'no'; stop; end;
      Case2: if FLAG=0 and stack=1 %% inf descending seq not found
          then begin; output 'yes'; stop; end;
      Case3: if FLAG=0 and length-stack > 1 %% top element cannot be continued infinitely
          then begin; %% try next
          pop N; push N+1; FLAG:=1; %% flag that fresh element is put on stack
          goto Loop;
          end;
      Case4: if FLAG=1 and stack-is-decreasing
          then begin;
          push 0; %% try to continue sequence with 0
          FLAG:=0; FLAG:=1; %% flash the flag
          goto Loop;
          end;
      Case5: if FLAG=1 and not stack-is-decreasing
          then begin;
          pop N; push N+1; %% try next
          FLAG:=0; FLAG:=1; %% flash the flag
          goto Loop;
          end;
```

Lemma 2. *Let* $I: \theta \to \omega$, $R: \theta \to ({}^{\omega}\omega)$ *be the computation by* P *with oracle* Z *and trivial input* $(0, 0, \ldots)$*. Then*

a) *If* Z *is wellfounded then the computation stops with output* `'yes'`*.*

b) *If* Z *is illfounded then the computation stops with output* `'no'`*.*

Theorem 3. *The set* $\mathrm{WO} = \{Z \subseteq \omega \,|\, Z \textit{ codes a wellorder}\}$ *is computable by an ITRM.*

Theorem 4. *Every* Π^1_1 *set* $A \subseteq \mathcal{P}(\omega)$ *is ITRM-computable.*

ITTMs can simulate ITRMs:

Simulate the number i in register R_m as an initial segment of i 1's on the m-th tape of an ITTM.

If λ is a limit time and $\liminf_{\tau\to\lambda} R_m(\tau) = i^* \leqslant \omega$ then the m-th tape will hold an initial segment of i^* 1's.

OK, if i^* is finite.

If $i^* = \omega$, this may be detected by a subroutine which then *resets* the m-th tape to 0.

Since ITTM-decidable $\subsetneq \mathbf{\Delta}^1_2$:

Ordinal register computability

Register machines	space ω	space admissible α	space Ord
time ω	standard register machine computable $= \Delta^0_1$	-	-
time admissible α	?	α register machine (α recursion theory) computable $= \boldsymbol{\Delta}_1(L_\alpha)$	-
time Ord	ITRM $\boldsymbol{\Delta}^1_1 \subsetneq$ computable in real parameter $\subsetneq \boldsymbol{\Delta}^1_2$	?	Ordinal register machine computable $= L \cap \mathcal{P}(\mathrm{Ord})$