

Checking Natural Language Proofs

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Meeting in Honor of Jouko Väänänen's Sixtieth Birthday

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- Proofs and formal derivations
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What is a mathematical proof?

What is a mathematical proof?

Mathematical logic:

A proof is a formal derivation in a (first-order) calculus

The Gödel completeness theorem



(Doctoral Dissertation, Vienna 1929)

The Gödel completeness theorem

$$\frac{\Gamma \quad \varphi}{\Gamma \quad \psi \quad \varphi} ; \quad \frac{}{\Gamma \quad \varphi \quad \varphi} ; \quad \frac{\Gamma \quad \varphi \quad \psi}{\Gamma \quad \varphi \rightarrow \psi} ; \quad \frac{\Gamma \quad \varphi}{\Gamma \quad \psi} ; \quad \frac{\Gamma \quad \varphi}{\Gamma \quad \neg \varphi} ;$$

$$\frac{\Gamma \quad \neg \varphi \quad \perp}{\Gamma \quad \varphi} ; \quad \frac{\Gamma \quad \varphi \frac{y}{x}}{\Gamma \quad \forall x \varphi} , \text{ if } y \notin \text{free}(\Gamma \cup \{\forall x \varphi\}); \quad \frac{\Gamma \quad \forall x \varphi}{\Gamma \quad \varphi \frac{t}{x}} ;$$

$$\frac{}{\Gamma \quad t \equiv t} ; \quad \frac{\Gamma \quad \varphi \frac{t}{x} \quad \Gamma \quad t \equiv t'}{\Gamma \quad \varphi \frac{t'}{x}}$$

The Gödel completeness theorem

$$\frac{\Gamma \quad \varphi}{\Gamma \quad \psi \quad \varphi} ; \frac{}{\Gamma \quad \varphi \quad \varphi} ; \frac{\Gamma \quad \varphi \quad \psi}{\Gamma \quad \varphi \rightarrow \psi} ; \frac{\Gamma \quad \varphi}{\Gamma \quad \psi} ; \frac{\Gamma \quad \varphi}{\Gamma \quad \neg \varphi} ;$$

$$\frac{\Gamma \quad \neg \varphi \quad \perp}{\Gamma \quad \varphi} ; \frac{\Gamma \quad \varphi \frac{y}{x}}{\Gamma \quad \forall x \varphi}, \text{ if } y \notin \text{free}(\Gamma \cup \{\forall x \varphi\}); \dots$$

$$\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x \equiv y) \dots$$

$$\forall x_1 \dots \forall x_n \forall x \exists y \forall z (z \in y \leftrightarrow z \in x \wedge \varphi(z, x_1, \dots, x_n)) \dots$$

$$\exists x (\exists y (y \in x \wedge \forall z \neg z \in y) \wedge \forall y (y \in x \rightarrow \exists z (z \in x \wedge \forall w (w \in z \leftrightarrow w \in y \vee w \equiv y)))) \dots$$

The Gödel completeness theorem

$$\frac{\Gamma \quad \varphi}{\Gamma \quad \psi \quad \varphi} ; \quad \frac{}{\Gamma \quad \varphi \quad \varphi} ; \quad \frac{\Gamma \quad \varphi \quad \psi}{\Gamma \quad \varphi \rightarrow \psi} ; \quad \frac{\Gamma \quad \varphi}{\Gamma \quad \psi} ; \quad \frac{\Gamma \quad \varphi}{\Gamma \quad \neg \varphi} ;$$

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Every logically true mathematical statement has a formal derivation.

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Every true mathematical statement has a formal derivation within some (foundational) axiom system.

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Every mathematical proof can be replaced by a formal derivation.

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Every true mathematical statement has a formal derivation within some (foundational) axiom system.

Every mathematical proof can be replaced by a formal derivation.

Mathematics can be in principle be carried out completely formal (Formal mathematics).

Pythagoras theorem



Pythagoras theorem



Graphical proofs?

Graphical proofs?

63=65?



Theorem. In a right-angled triangle (ABC) the square on the hypotenuse (AB) is equal to the sum of the squares on the other two sides (AC, BC) .



Dem.---On the sides AB , BC , CA describe squares [xlvi.]. Draw CL parallel to AG . Join CG , BK . Then because the angle ACB is right (hyp.), and ACH is right, being the angle of a square, the sum of the angles ACB , ACH is two right angles; therefore BC , CH are in the same right line [xiv.]. In like manner AC , CD are in the same right line. Again, because BAG is the angle of a square it is a right angle: in like manner CAK is a right angle. Hence BAG is equal to CAK : to each add BAC , and we get the angle CAG equal to KAB . Again, since BG and CK are squares, BA is equal to AG , and CA to AK . Hence the two triangles CAG , KAB have the sides CA , AG in one respectively equal to the sides KA , AB in the other, and the contained angles CAG , KAB also equal. Therefore [iv.] the triangles are equal; but the parallelogram AL is double of the triangle CAG [xli.], because they are on the same base AG , and between the same parallels AG and CL . In like manner the parallelogram AH is double of the triangle KAB , because they are on the same base AK , and between the same parallels AK and BH ; and since doubles of equal things are equal (Axiom vi.), the parallelogram AL is equal to AH . In like manner it can be proved that the parallelogram BL is equal to BD . Hence the whole square AF is equal to the sum of the two squares AH and BD .

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1.	Φ_{Gr}	$\neg \circ v_0 e \equiv v_0$		$\neg \exists v_0 \neg \circ v_0 e \equiv v_0$	VR
2.	Φ_{Gr}	$\neg \circ v_0 e \equiv v_0$		$\neg \circ v_0 e \equiv v_0$	VR
3.	Φ_{Gr}	$\neg \circ v_0 e \equiv v_0$		$\exists v_0 \neg \circ v_0 e \equiv v_0$	$\exists S$ auf 2
4.	Φ_{Gr}			$\circ v_0 e \equiv v_0$	WR auf 1,3
5.				$(v_2 \equiv \circ v_0 e) \frac{\circ v_0 e}{v_2}$	(\equiv)
6.		$\circ v_0 e \equiv v_0$		$(v_2 \equiv \circ v_0 e) \frac{v_0}{v_2}$	Sub auf 5
7.	Φ_{Gr}	$\circ v_0 e \equiv v_0$		$v_0 \equiv \circ v_0 e$	AR auf 6
8.	Φ_{Gr}			$v_0 \equiv \circ v_0 e$	KS auf 4,7
9.	Φ_{Gr}	$v_0 \equiv e$		$v_0 \equiv e$	VR
10.	Φ_{Gr}	$v_0 \equiv e$		$(\neg \circ v_0 e \equiv e \vee v_0 \equiv e)$	$\vee S$ auf 9
11.	Φ_{Gr}	$\neg v_0 \equiv e$		$(\neg v_2 \equiv e) \frac{v_0}{v_2}$	VR
12.	Φ_{Gr}	$\neg v_0 \equiv e$	$v_0 \equiv \circ v_0 e$	$(\neg v_2 \equiv e) \frac{\circ v_0 e}{v_2}$	Sub auf 11
13.	Φ_{Gr}	$\neg v_0 \equiv e$	$v_0 \equiv \circ v_0 e$	$\neg \circ v_0 e \equiv e$	12
14.	Φ_{Gr}	$\neg v_0 \equiv e$		$v_0 \equiv \circ v_0 e$	AR auf 8
15.	Φ_{Gr}	$\neg v_0 \equiv e$		$\neg \circ v_0 e \equiv e$	KS auf 14
16.	Φ_{Gr}	$\neg v_0 \equiv e$		$(\neg \circ v_0 e \equiv e \vee v_0 \equiv e)$	$\vee S$ auf 15
17.	Φ_{Gr}			$(\neg \circ v_0 e \equiv e \vee v_0 \equiv e)$	FU auf 10,16
18.	Φ_{Gr}	$\neg(\neg \circ v_0 e \equiv e \vee v_0 \equiv e)$	$\neg \neg \exists v_0 \neg(\neg \circ v_0 e \equiv e \vee v_0 \equiv e)$	$(\neg \circ v_0 e \equiv e \vee v_0 \equiv e)$	AR auf 17
19.	Φ_{Gr}	$\neg(\neg \circ v_0 e \equiv e \vee v_0 \equiv e)$	$\neg \neg \exists v_0 \neg(\neg \circ v_0 e \equiv e \vee v_0 \equiv e)$	$\neg(\neg \circ v_0 e \equiv e \vee v_0 \equiv e)$	VR
20.	Φ_{Gr}	$\neg(\neg \circ v_0 e \equiv e \vee v_0 \equiv e)$		$\neg \exists v_0 \neg(\neg \circ v_0 e \equiv e \vee v_0 \equiv e)$	WR auf 18,19
21.	Φ_{Gr}	$\exists v_0 \neg(\neg \circ v_0 e \equiv e \vee v_0 \equiv e)$		$\neg \exists v_0 \neg(\neg \circ v_0 e \equiv e \vee v_0 \equiv e)$	$\exists A$ auf 20
22.	Φ_{Gr}	$\neg \exists v_0 \neg(\neg \circ v_0 e \equiv e \vee v_0 \equiv e)$		$\neg \exists v_0 \neg(\neg \circ v_0 e \equiv e \vee v_0 \equiv e)$	VR
23.	Φ_{Gr}			$\neg \exists v_0 \neg(\neg \circ v_0 e \equiv e \vee v_0 \equiv e)$	FU auf 21,22

1. $\Phi_{Gr} \quad \neg \circ v_0 e \equiv v_0$ $\neg \exists v_0 \neg \circ v_0 e \equiv v_0$ VR
2. $\Phi_{Gr} \quad \neg \circ v_0 e \equiv v_0$ $\neg \circ v_0 e \equiv v_0$ VR
3. $\Phi_{Gr} \quad \neg \circ v_0 e \equiv v_0$ $\exists v_0 \neg \circ v_0 e \equiv v_0$ $\exists S$ auf 2
4. Φ_{Gr} $\circ v_0 e \equiv v_0$ WR auf 1,3

- | | | | | |
|----|-------------|-------------------------------|------------------------------------------------|-------------------|
| 1. | Φ_{Gr} | $\neg \circ v_0 e \equiv v_0$ | $\neg \exists v_0 \neg \circ v_0 e \equiv v_0$ | VR |
| 2. | Φ_{Gr} | $\neg \circ v_0 e \equiv v_0$ | $\neg \circ v_0 e \equiv v_0$ | VR |
| 3. | Φ_{Gr} | $\neg \circ v_0 e \equiv v_0$ | $\exists v_0 \neg \circ v_0 e \equiv v_0$ | $\exists S$ auf 2 |
| 4. | Φ_{Gr} | | $\circ v_0 e \equiv v_0$ | WR auf 1,3 |
| | | | \vdots | |

- | | | | |
|----|-----------------------------------------|------------------------------------------------|-------------------|
| 1. | $\Phi_{Gr} \neg \circ v_0 e \equiv v_0$ | $\neg \exists v_0 \neg \circ v_0 e \equiv v_0$ | VR |
| 2. | $\Phi_{Gr} \neg \circ v_0 e \equiv v_0$ | $\neg \circ v_0 e \equiv v_0$ | VR |
| 3. | $\Phi_{Gr} \neg \circ v_0 e \equiv v_0$ | $\exists v_0 \neg \circ v_0 e \equiv v_0$ | $\exists S$ auf 2 |
| 4. | Φ_{Gr} | $\circ v_0 e \equiv v_0$ | WR auf 1,3 |
| | | \vdots | |

- | | | | | |
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| 1. | Φ_{Gr} | $\neg \circ v_0 e \equiv v_0$ | $\neg \exists v_0 \neg \circ v_0 e \equiv v_0$ | VR |
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| | | | \vdots | |

21. Φ_{Gr} $\exists v_0 \neg(\neg \circ v_0 e \equiv e \vee v_0 \equiv e)$ $\neg \exists v_0 \neg(\neg \circ v_0 e \equiv e \vee v_0 \equiv e)$ $\exists A$ auf 20
22. Φ_{Gr} $\neg \exists v_0 \neg(\neg \circ v_0 e \equiv e \vee v_0 \equiv e)$ $\neg \exists v_0 \neg(\neg \circ v_0 e \equiv e \vee v_0 \equiv e)$ VR
23. Φ_{Gr} $\neg \exists v_0 \neg(\neg \circ v_0 e \equiv e \vee v_0 \equiv e)$ FU auf 21,22

Formal proofs - derivations

$$\frac{\Gamma \quad \varphi}{\Gamma \quad \psi \quad \varphi}$$

$$\frac{\Gamma \quad \varphi}{\Gamma \quad \neg \varphi}$$

$$\frac{\Gamma \quad \neg \varphi}{\Gamma \quad \perp}$$

$$\frac{}{\Gamma \quad t \equiv t}$$

$$\frac{\Gamma \quad \varphi \quad \varphi}{\Gamma \quad \varphi \quad \psi}$$

$$\frac{\Gamma \quad \varphi \quad \psi}{\Gamma \quad \varphi \rightarrow \psi}$$

$$\frac{\Gamma \quad \neg \varphi \quad \perp}{\Gamma \quad \varphi}$$

$$\frac{\Gamma \quad \varphi \frac{t}{x}}{\Gamma \quad t \equiv t'}$$

$$\frac{\Gamma \quad t \equiv t'}{\Gamma \quad \varphi \frac{t'}{x}}$$

$$\frac{\Gamma \quad \varphi}{\Gamma \quad \varphi \rightarrow \psi}$$

$$\frac{\Gamma \quad \varphi \rightarrow \psi}{\Gamma \quad \psi}$$

$$\frac{\Gamma \quad \varphi \frac{y}{x}}{\Gamma \quad \forall x \varphi},$$

if $y \notin \text{free}(\Gamma \cup \{\forall x \varphi\})$

$$\frac{\Gamma \quad \varphi}{\Gamma \quad \neg \varphi}$$

$$\frac{\Gamma \quad \neg \varphi}{\Gamma \quad \perp}$$

$$\frac{\Gamma \quad \forall x \varphi}{\Gamma \quad \varphi \frac{t}{x}}$$

Formal proofs - derivations

- derivations are formed by repeated applications of (simple) syntactic rules
- whether a formal text is a derivation can (easily) be checked algorithmically

Formal proofs - derivations

N. Bourbaki:

If formalized mathematics were as simple as the game of chess, then once our chosen formalized language had been described there would remain only the task of writing out our proofs in this language, [...] But the matter is far from being as simple as that, and no great experience is necessary to perceive that such a project is absolutely unrealizable: the tiniest proof at the beginnings of the Theory of Sets would already require several hundreds of signs for its complete formalization. [...] formalized mathematics cannot in practice be written down in full, [...] We shall therefore very quickly abandon formalized mathematics, [...]

Formal proofs

Saunders Mac Lane:

As to precision, we have now stated an absolute standard of rigor: A mathematical proof is rigorous when it is (or could be) written out in the first-order predicate language $L(\in)$ as a sequence of inferences from the axioms ZFC, each inference made according to one of the stated rules. [...] When a proof is in doubt, its repair is usually a partial approximation to the fully formal version.

Computer-supported formal proofs

J. McCarthy:

Checking mathematical proofs is potentially one of the most interesting and useful applications of automatic computers. ... Proofs to be checked by computer may be briefer and easier to write than the informal proofs acceptable to mathematicians. This is because the computer can be asked to do much more work to check each step than a human is willing to do, and this permits longer and fewer steps.

McCarthy, J. "Computer Programs for Checking Mathematical Proofs," Proceedings of the Symposium in Pure Math, Recursive Function Theory, Volume V, pages 219-228, AMS, Providence, RI, 1962.

Automatic proof checker

Automath (~1967)

N.G. de Bruijn

From the [Automath](#) formalization of E. Landau, *Grundlagen der Analysis*, 1930
by L. S. van Benthem Jutting, 1979:

```

ic:=pli(0,1rl):complex
+10300
t1:=tsis12a(0,1rl,0,1rl):is(ts(ic,ic),pli(mn"r"(ts"r"(0,0),ts"r"(1rl,1rl)),pl"r"(ts"r"(0,1rl),
ts"r"(1rl,0))))
t2:=tris(real,mn"r"(ts"r"(0,0),ts"r"(1rl,1rl)),m0"r"(ts"r"(1rl,1rl)),m0"r"(1rl),pl01(ts"r"(0,0),
m0"r"(ts"r"(1rl,1rl)),ts01(0,0,refis(real,0))),ism0"r"(ts"r"(1rl,1rl),1rl,satz195(1rl))):
is"r"(mn"r"(ts"r"(0,0),ts"r"(1rl,1rl)),m0"r"(1rl))
t3:=tris(real,pl"r"(ts"r"(0,1rl),ts"r"(1rl,0)),ts"r"(1rl,0),0,pl01(ts"r"(0,1rl),ts"r"(1rl,0),
ts01(0,1rl,refis(real,0))),ts02(1rl,0,refis(real,0))):is"r"(pl"r"(ts"r"(0,1rl),ts"r"(1rl,0)),0)
t4:=isrecx12(mn"r"(ts"r"(0,0),ts"r"(1rl,1rl)),m0"r"(1rl),pl"r"(ts"r"(0,1rl),
ts"r"(1rl,0)),0,t2,t3):is(pli(mn"r"(ts"r"(0,0),ts"r"(1rl,1rl)),
pl"r"(ts"r"(0,1rl),ts"r"(1rl,0))),cofrl(m0"r"(1rl)))
t5:=satz298j(1rl):is(cofrl(m0"r"(1rl)),m0(1c))
-10300
satz2300:=tr3is(cx,ts(ic,ic),pli(mn"r"(ts"r"(0,0),ts"r"(1rl,1rl)),
pl"r"(ts"r"(0,1rl),ts"r"(1rl,0))),cofrl(m0"r"(1rl)),m0(1c),t1".10300",t4".10300",t5".10300"):
is(ts(ic,ic),m0(1c))

```

The **MIZAR** system (1973 -) of Andrzej Trybulec

Language modeled after
“mathematical vernacular”

Natural deduction style

Automatic proof checker

Large mathematical library

Journal

Formalized Mathematics

www.mizar.org



MIZAR example: Proof of Pythagoras

```

theorem for p1,p2,p3 st p1<>p2 & p3<>p2 &
  (angle(p1,p2,p3)=PI/2 or angle(p1,p2,p3)=3/2*PI) holds
  (|.p1-p2.|^2+|.p3-p2.|^2=|.p1-p3.|^2
  proof let p1,p2,p3; assume A1: p1<>p2 & p3<>p2 &
    (angle(p1,p2,p3)=PI/2 or angle(p1,p2,p3)=3/2*PI);
  then A2: euc2cpx(p1)<> euc2cpx(p2) by Th6;
  A3: euc2cpx(p3)<> euc2cpx(p2) by A1,Th6;
  A4: euc2cpx(p1) - euc2cpx(p2) = euc2cpx(p1-p2) by Th19;
  A5: euc2cpx(p3) - euc2cpx(p2) = euc2cpx(p3-p2) by Th19;
  A6: euc2cpx(p1) - euc2cpx(p3) = euc2cpx(p1-p3) by Th19;
  A7: angle(p1,p2,p3) = angle(euc2cpx(p1), euc2cpx(p2), euc2cpx(p3))
by Def4;
  A8: |.euc2cpx(p1-p2).| = |.p1-p2.| by Th31;
  A9: |.euc2cpx(p3-p2).| = |.p3-p2.| by Th31;
  |.euc2cpx(p1-p3).| = |.p1-p3.| by Th31;
  hence thesis by A1,A2,A3,A4,A5,A6,A7,A8,A9,COMPLEX2:91;
end;

```

Derivations

- (Euclid)
- (Hilbert-style) calculi
- Automath
- MIZAR

What is a mathematical proof?

- description of the/some mathematical “reality”?
- argumentative text about the/some mathematical “reality”?
- argumentative text within some system of initial assumptions (axioms)?
- Wittgenstein: ?
- abbreviation for some (long) formal derivation?
- recipe for building a formal derivation if required?
- a formal derivation in some very rich formal system (Montague: English as a formal language)?

Jody Azzouni: The derivation-indicator view of mathematical practice

ABSTRACT. A version of Formalism is vindicated: Ordinary mathematical proofs indicate (one or another) mechanically checkable derivation of theorems from the assumptions those ordinary mathematical proofs presuppose. The indicator view explains why mathematicians agree so readily on results established by proofs in ordinary language that are (palpably) not mechanically checkable. Mechanically checkable derivations in this way structure ordinary mathematical practice without its being the case that ordinary mathematical proofs can be 'reduced to' such derivations. In this way, one threat to formalist-style positions is removed: Platonic objects aren't needed to explain how mathematicians understand the import of ordinary mathematical proofs. (*Philosophia Mathematica*, 2004)

Derivation-indication

N. Bourbaki:

If formalized mathematics were as simple as the game of chess, ...

... there would remain only the task of **writing out** our proofs in this language, ...

Derivation-indication

Saunders Mac Lane:

As to precision, we have now stated an absolute standard of rigor: A mathematical proof is rigorous when it is (or could be) **written out** in the first-order predicate language $L(\in)$ as a sequence of inferences from the axioms ZFC, each inference made according to one of the stated rules. [...] When a proof is in doubt, its repair is usually a **partial approximation** to the fully formal version.

Derivation-indication

- Mathematicians agree that proofs can be **written out** in increasingly formal detail
- This leads to a fully formal derivation after some (long) finite time
- The indicator function lies mainly in the natural language parts of proofs
- Can one identify indicators by natural language processing?
- Derivations may be derivations performed by an Automatic Theorem Prover (ATP).

The **Naproche** project: **Natural language proof checking**

- studies the syntax and semantics of the language of proofs, emphasizing natural language and natural argumentation aspects
- models natural language proofs using computer-supported methods of formal linguistics and formal logic
- “reverse engineering” approach to derivation-indication
- joint work with Bernhard Schröder, linguistics; Bonn, Essen, Cologne; www.naproche.net
- development of a mathematical authoring system with a L^AT_EX-quality graphical interface

The **Naproche** project: **Natural language proof checking**

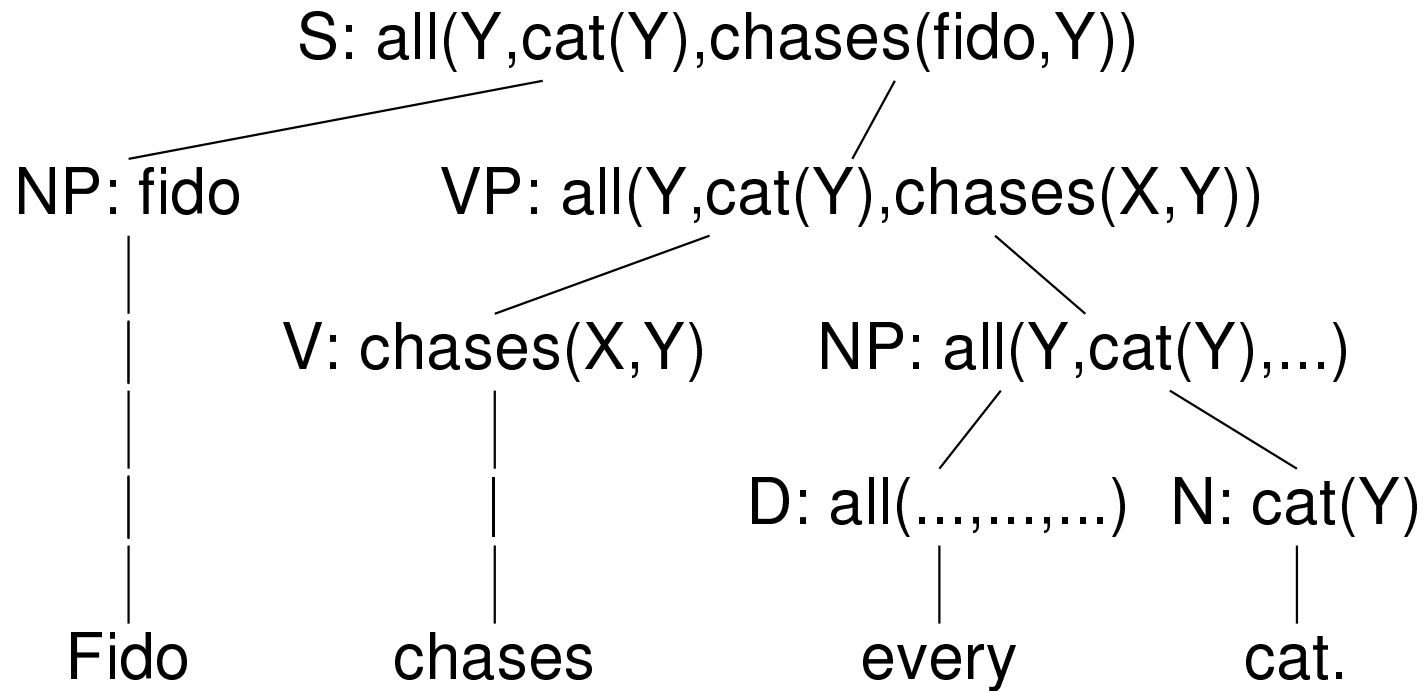
- To devise a strictly formal system for mathematics, implemented by computer, whose input language is an extensive part of the common mathematical language, and whose proof style is close to proof styles found in the mathematical literature.

Mathematical statements

“1 divides every integer.” \longleftrightarrow “Fido chases every cat.”

Linguistic analysis

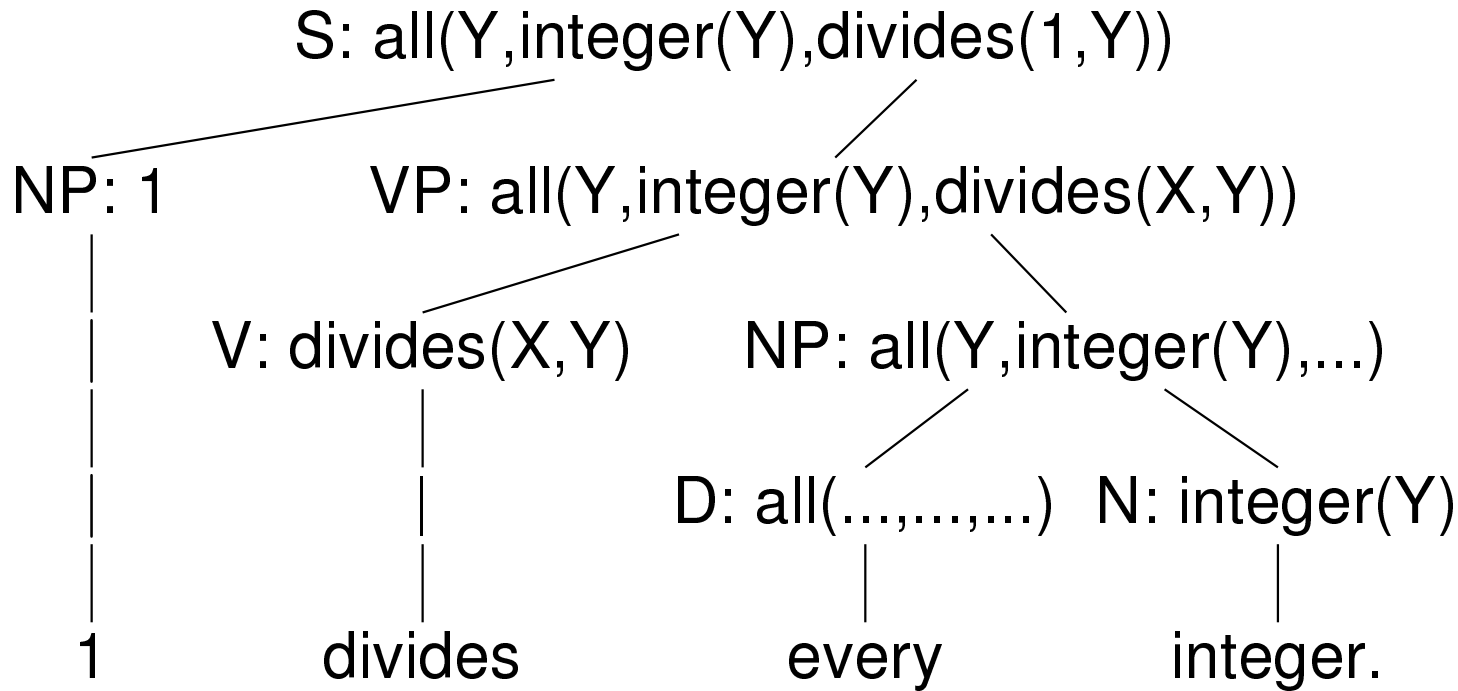
“Fido chases every cat.”



$\forall Y (\text{cat}(Y) \rightarrow \text{chases}(\text{fido}, Y)).$

Linguistic analysis

“1 divides every integer.”



$\forall Y (\text{integer}(Y) \rightarrow 1|Y).$

Layers of the **Naproche system**:

↓ Standard or web editor

TeX-style input text

↕ Natural language processing (NLP)

Proof representation structure (PRS)

↕ First-order translation

First-order logic format (TPTP)

↕ Proof checker or automatic theorem prover
(ATP)

“Accepted”/“Not accepted”, with error messages

E. Landau, *Grundlagen der Analysis*, 1930: Theorem 30



Components of the Naproche system: linguistic analysis

- standard analysis by a Prolog Definite Clause Grammar (DCG), the grammar defines a controlled natural language for mathematics (CNL), i.e. a formal subset of the common mathematical language
- translation into a formal semantics (without ambiguity)

Components of the Naproche system: linguistic analysis

- formal semantics: proof representation structures (PRS), extending discourse representation structures (DRS)
- DRS: tool for anaphor resolution (Let x be a set. It is ...)
and for interpretation of natural language quantification
(Every prime number is positive; a prime number is positive)
- PRS, moreover, represent global text structurings: Theorem / Proof, introductions and retractions of assumptions

Components of the Naproche system: Checking logical correctness

- translating the PRS conditions into some first-order format
- use TPTP-format (Thousands of Problems for Theorem Provers)
- generate relevant premises for every condition
- automatic theorem prover (ATP) used to prove every condition from its relevant premises
- proof is accepted if ATP can prove every condition
- feedback of success/error messages

Results

- The Naproche system allows natural reformulation of (simple) mathematical texts
- some example texts and parts of Landau, Foundations of Analysis have been reformulated and checked

Chapter 1 from Landau in Naproche

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Abstract

This is a reformulation of the first chapter of Landau's *Grundlagen der Analysis* in the Controlled Natural Language of Naproche. Talk about sets is still avoided. One consequence of this is that Axiom 5 (the induction axiom) cannot be formulated; instead we use an induction proof method.

Axiom 3: For every x , $x' \neq 1$.

Axiom 4: If $x' = y'$, then $x = y$.

Theorem 1: If $x \neq y$ then $x' \neq y'$.

Proof:

Assume that $x \neq y$ and $x' = y'$. Then by axiom 4, $x = y$. Qed.

Theorem 2: For all x $x' \neq x$.

Proof:

By axiom 3, $1' \neq 1$. Suppose $x' \neq x$. Then by theorem 1, $(x')' \neq x'$. Thus by induction, for all x $x' \neq x$. Qed.

Theorem 3: If $x \neq 1$ then there is a u such that $x = u'$.

Proof:

If $1 \neq 1$ then there is a u such that $1 = u'$.

Assume $x' \neq 1$. If $u = x$ then $x' = u'$. So there is a u such that $x' = u'$.

Thus by induction, if $x \neq 1$ then there is a u such that $x = u'$. Qed.

Definition 1:

Define $+$ recursively:

$$x + 1 = x'.$$

$$x + y' = (x + y)'.$$

Theorem 5: For all x, y, z , $(x + y) + z = x + (y + z)$.

Proof:

Fix x, y .

$$(x + y) + 1 = (x + y)' = x + y' = x + (y + 1).$$

Assume that $(x + y) + z = x + (y + z)$. Then $(x + y) + z' = ((x + y) + z)' = (x + (y + z))' = x + (y + z)' = x + (y + z')$. So $(x + y) + z' = x + (y + z')$.

Thus by induction, for all z , $(x + y) + z = x + (y + z)$. Qed.

Lemma 4a: For all y , $1 + y = y'$.

Proof:

By definition 1, $1 + 1 = 1'$.

Suppose $1 + y = y'$. Then by definition 1, $1 + y' = (1 + y)'$. So $1 + y' = (y')'$.

Thus by induction, for all y $1 + y = y'$. Qed.

Possible applications

- Natural language interfaces to formal mathematics
- Mathematical authoring and checking tools
- writing texts that are simultaneously acceptable by human readers and formal mathematics systems (“Logic for men and machines”)
- Tutorial applications: teaching how to prove

General issues

- Naproche attempts to implement parts of the derivation-indication approach to proofs
- natural language components serve as indicators
- there are natural(ly looking) proofs that are fully formal with respect to the Naproche system
- this defines a “fortified formalism”, using linguistic methods and computer implementations, which allows to view some natural proofs as fully formal
- can a “fortified formalism” help to mediate between the “two streams” in the philosophy of mathematics (formalistic / naturalistic)?

Thank You!