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E_n-Homology

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Overview

The notion of an operad emerged in algebraic topology in May's study of iterated loopspaces [May]. Operads model properties of operations such as associativity and commutativity. The operations encoded in a given operad are realized by algebras over this operad.

Let dg-mod denote the category of differential graded modules over a fixed ground ring k. Examples of algebras over operads in dg-mod include A_∞-algebras and E_∞-algebras, generalizing the concept of associativity and commutativity:

- An A_∞-algebra is a differential graded module equipped with a multiplication that is associative up to homotopies of all possible higher degrees.
- An E_∞-algebra is a differential graded module with a multiplication which is associative and commutative up to all higher homotopies.

These algebras are algebras over so called E_n-operads, the notion of A_∞-algebras corresponding to the case $n = 1$, the example of E_∞-algebras corresponding to $n = \infty$.

To every (sufficiently good) operad $\mathtt{O}$ we can associate a homology theory $H_*^{\mathtt{O}}$ and a cohomology theory $H^*_{\mathtt{O}}$ defined for algebras over the operad $\mathtt{O}$. In particular, we can associate a homology and a cohomology theory to E_n-operads.

By neglect of structure every ordinary commutative associative algebra A can be thought of as an algebra over an E_n-operad. In particular, the homology and cohomology theories associated to the operad are defined for A.

The cases $n = 1$ and $n = \infty$ once again provide familiar examples:
For $n = 1$ we retrieve Hochschild homology and cohomology, the classical theory associated to associative algebras. For $n = \infty$ the homology and cohomology associated to E_∞-algebras coincides with André-Quillen homology and cohomology if applied to ordinary commutative associative algebras.

The project described here is concerned with gaining knowledge about the intermediate cases $1 < n < \infty$ by constructing and investigating additional structure of E_n-cohomology of commutative associative algebras.

Operads

Operads were introduced in 1972 by May in his work [May] on iterated loop spaces. We only give a rough definition:

Definition 1 *An operad $\mathtt{O}$ consists of a collection $(\mathtt{O}(j))_{j\geq 0}$ of objects in a given symmetric monoidal category, so that each $\mathtt{O}(j)$ is endowed with an action of the symmetric group Σ_j, and of morphisms*

$$\gamma_{s,r_1,\dots,r_s}\colon \mathtt{O}(s)\otimes\mathtt{O}(r_1)\otimes\dots\otimes\mathtt{O}(r_s)\to\mathtt{O}(r_1+\dots+r_s),$$

for every choice of $s, r_1, \dots, r_s \geq 0$, such that certain associativity and unit axioms are satisfied.

The intuition behind this concept is that operads encode operations: Roughly speaking, the objects $\mathtt{O}(j)$ consist of operations with j inputs and one output, and the morphisms $\gamma_{s,r_1,\dots,r_s}$ describe the new operation we get if we substitute the s inputs of a given s-ary operation by the outputs of s other given operations.

The prototypical model for the structure of an operad is therefore the endomorphism operad End_X for a given object X in a symmetric monoidal category, with

$$End_X(j) := Mor(X^{\otimes j}, X),$$

where the symmetric group acts by permuting the factors of $X^{\otimes j}$ and the morphisms γ correspond to substitution of arguments as described above.

Definition 2 *An object X in the given symmetric monoidal category is called an algebra over $\mathtt{O}$, if it realizes the operations abstractly encoded in the operad, i.e. if there is a morphism of operads $\mathtt{O}\to End_X(j)$.*

For example, there is an operad Ass in the category of modules over a ground ring k, so that being an associative k-algebra is equivalent to being an algebra over Ass. Similarly, there is an algebra Com encoding commutative algebras.

May used the *little n-cubes operad* $\mathtt{C}_n$, an operad in the category of topological spaces, in [May] to generalize a result from Boardman and Vogt [BV], proving a recognition principle for iterated loop spaces:

Theorem 1 *(May) Let Y be a connected CW complex with nondegenerate basepoint, $1 \leq n \leq \infty$. If Y is an algebra over $\mathtt{C}_n$, there is a topological space X with $Y \sim \Omega^n X$.*

This result holds for a greater class of operads: Instead of the little n-cubes operad one can consider any E_n-operad, an operad equivalent in some sense to the little n-cubes operad.

Commutative Algebras as E_n-Algebras and their Homology

Commutative Algebras as E_n-Algebras

There is a variant of the notion of an E_n-operad in the category dg-mod of chain complexes. For $n = 1$ and $n = \infty$ one retrieves the notions of A_∞- and E_∞-algebras, generalizing the notions of associativity and commutativity.
An obvious example of an E_∞-algebra is given by any strictly commutative and associative algebra over the ground ring.

One can think of algebras over E_n-operads with arbitrary n as some kind of interpolation between these two cases. In particular, every ordinary commutative associative algebra is an E_∞-algebra and therefore an algebra over some E_n-operad for any other n as well.

E_n-Homology

To every E_n-operad one can define a homology and a cohomology theory. Considering a commutative associative algebra as an algebra over an E_n-operad, this particularly defines a homology and cohomology theory for commutative and associative algebras, retrieving familiar theories in the cases $n = 1$ and $n = \infty$ as described above.

Recent developments provide new approaches to E_n-homology. In [F] Fresse generalizes the construction of the classical bar complex of differential graded algebras to E_n-algebras. He finds that one can iterate this construction n times for a given E_n-algebra and that the iterated bar complex serves as some sort of delooping:

Theorem 2 *(Fresse) If A is a (sufficiently good) algebra over an (sufficiently good) E_n-operad $\mathtt{E}_n$, the $\mathtt{E}_n$-homology of A equals the homology of the n-th desuspension of the n-th iterated bar complex of the chain complex A:*

$$H_*^{\mathtt{E}_n}(A) \cong H_*(\Sigma^{-n}B^n(A)).$$

He also shows that in the case of an ordinary commutative associative algebra the operadic bar construction coincides with the classical bar construction of Eilenberg and Mac Lane [EM].

In [LR] Livernet and Richter observe that E_n-homology of commutative associative algebras may be interpreted as the homology of certain functors. More precisely, they define the E_n-homology of functors $F\colon \mathrm{Epi}_n \to k{-}\mathrm{mod}$ from the category Epi_n of planar trees with n levels to the category of k-modules by constructing an explicit multicomplex associated to F.
They further show that one can calculate the homology of these functors via derived functors:

Theorem 3 *(Livernet-Richter) There exists a functor $b_n^{epi}\colon Epi_n \to k{-}\mathrm{mod}$ so that for every functor $F\colon Epi_n \to k{-}\mathrm{mod}$ one has*

$$H_*^{\mathtt{E}_n}(F) \cong Tor_*^{Epi_n}(b_n^{epi}, F)$$

Additional Structure

The project described is mainly concerned with investigating two possible additional structures on E_n-cohomology:

- For $n = 1$ and $n = \infty$, the associated cohomology theories, i.e. Hochschild and André-Quillen cohomology, are equipped with a well known graded lie bracket.
  One aim of this project is to investigate if the approaches to E_n-cohomology described above provide an opportunity to endow E_n-cohomology with a graded lie bracket for arbitrary n.
- Objects expressed via Ext functors naturally allow the definition of cohomology operations. One aim of this project is to construct and understand cohomology operations on E_n-cohomology by utilizing an interpretation of E_n-cohomology in terms of Ext functors similar to the results in [LR].

References

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