Graduate Student Seminar
Summer Term 2009:
Nilpotence in Stable Homotopy Theory

We start with a concrete question. Let X be a finite CW-complex and $f : \Sigma^d X \to X$ a map from the d-th suspension of X to X itself. We can iterate this map and get $f^{(k)} : \Sigma^{dk} X \to X$.

Question 1. *Is there a method to detect whether f is stably nilpotent in the sense that there is some $k > 0$ such that some suspension of $f^{(k)}$ is nullhomotopic? Are there $f : \Sigma^d X \to X$ for $d > 0$ which are not stably nilpotent?*

The first question in its many variants will be the main topic of the seminar, while we will touch only briefly upon the latter at the end. The first idea of an algebraic topologist to solve the first question is to feed the map f into a homology theory h_*, because, if f is stably nilpotent, we have that $h_*(f)$ is nilpotent, which might be easier to compute. A very suitable homology theory for this purpose is MU_*, the bordism theory of stably complex manifolds. There we have the following converse:

Theorem 1. *If $MU_*(f) = 0$, then f is stably nilpotent.*

While this form of the nilpotence theorem has the advantage of being quite elementary to state, it is not the form which is most suitable for a proof. Recall that a ring spectrum is just a monoid in the stable homotopy category and its homotopy groups get a ring structure by the multiplication map. We will deduce Theorem 1 from the following one, which might be called our *main theorem*:

Theorem 2. *Let R be a ring spectrum. Then every element of the kernel of the Hurewicz $\pi_*(R) \to MU_*(R)$ is nilpotent.*

This is a strong theorem. For example one easy application leads to a highly non-trivial result about the stable homotopy groups of spheres: We have that $MU_*(S^0) \cong \pi_*(MU) \cong \mathbb{Z}[x_0, x_1, \dots]$ is torsionfree. Since all homotopy groups $\pi_i(\mathbb{S}) = \pi_i^s(S^0)$ of the sphere spectrum are torsion for $i > 0$ the Hurewicz $\pi_i(\mathbb{S}) \to \pi_i(MU)$ is zero for all $i > 0$. So we get the following corollary:

Corollary 3 (Nishida)**.** *Every element of the stable homotopy groups of spheres of positive degree is nilpotent.*

The proof of Theorem 2 is quite difficult. Two of our main tools will be the Adams spectral sequence and the Snaith splitting, which are also of independent interest and will be presented in talks 3 and 6, respectively. Three talks will be spent on the proof of Theorem 2 itself. While the proof depends on the geometry of the spectrum MU, one can ask, if there are other spectra which detect nilpotency in the same way as MU does and if there is an algebro-topological criterion to find them. We will completely answer this question after constructing the spectra BP and the Morava K-theories $K(n)$.

The nilpotence theorem is only a small part of the Ravenel conjectures, first written up in [Rav84]. They provide a huge program of global and qualitative aspects of stable homotopy theory. All but one were proven in 1985 by Devinatz, Hopkins and Smith. A cornerstone of the proof of the Ravenel conjectures is the so called thick subcategory theorem which is a corollary of our nilpotence theorem. We will discuss these aspects in the last talk. For an amusing history of Ravenel's work by Hopkins, which mixes the mathematics with some jokes, see [Hop08].

Our main references are the original papers [DHS88] and [HS98] and the book [Rav92]. The latter provides more background material and a broader discussion of the Ravenel conjectures, but its proofs are often only sketchy. So most of the time we will rely on the original papers, which are well written.

The talks should be no longer than **75 minutes**. If you are short of time, you may skip some computations and technical details and concentrate on the ideas.

List of Talks

1. **The Stable Homotopy Category and Localization (23.4. - Jan Möllers):** Introduce the homotopy category of spectra and the concepts of connective spectra, ring spectra,... and their relation to homology theories (see e.g. [Rav92], 4.1 + A.2-4) and also mention the exact sequence of a cone (see [Ada74], III.3.9). Then give a short introduction to localization, completion and Bousfield classes ([Rav84]). Give the example of the sphere (in the unstable case) for localization at p.

2. **MU, Thom Spectra and the Nilpotence Theorems (23.4. -**

Achim Beckers): Define MU and more generally (complex) Thom spectra. Discuss shortly the relation to complex cobordism ([Rav92], B1-2). State the different versions of the nilpotence theorem ([DHS88], Thm 1(i)-(ii) + finite case of Corollary 2 + Thm 2) and prove their equivalence. For the needed Spanier-Whitehead-Duality see e.g. [Rav92], 5.2.

3. **The Adams Spectral Sequence (14.5. - Irakli Patchkoria):** Present the (generalized) Adams spectral sequence with its ingredients (Hopf algebroids, Adams resolutions, convergence,...). A short overview can be found in the appendix of [Rav92], a more thorough treatment in [Rav03] and [Ada74]. Time will not permit to give many proofs, but indicate as an application how to compute $\pi_*(MU)$ ([Rav03], Thm 3.1.5).

4. **Proof of Nilpotence Theorem I (14.5. - Boryana Dimitrova):** Define the spectra $X(n)$, give the plan of the proof and prove lemma 1 ([Rav92], 7.4, 9.1 and beginning of 9.2, or alternatively [DHS88], section 1).

5. **Proof of Nilpotence Theorem II (25.6. - Viktoriya Ozornova):** Prove lemma 2 via the Adams spectral sequence ([Rav92], 9.2, or [DHS88], section 2). Take the proof of the vanishing line of Ravenel.

6. **The Snaith Splitting (25.6 - Alexander Hess):** Prove the Snaith splitting as in [Böd87] (as far as time permits). With the same methods it should be possible to prove the James model. Do the reformulation in [Rav92], 9.4.

7. **Proof of Nilpotence Theorem III (9.7. - Lennart Meier):** The topic of this talk is the proof of lemma 3, the heart of the proof of the nilpotence theorem. The main source is [DHS88], section 3. A short version is presented in [Rav92], 9.5, for motivation see also 9.3 and 9.6. It will surely neither possible nor desirable to present all technical details (e.g. you should not talk about operads). It is more important to present and motivate the flow of ideas and show how the Snaith splitting comes in.

8. **BP and Morava K-Theory (9.7. - Moritz Groth):** Sketch and motivate the construction of BP as in [Wil82], §§1-3, or [Ada74], II.1-2 and 15-16. Then define Morava K-Theory (don't delve into the ring

structure) and prove some basic properties ([HS98], section 1 up to proposition 1.5).

9. **More Morava K-Theory (23.7. - Katja Hutschenreuter):** Prove the Morava K-theory versions of the nilpotence theorem ([HS98], Theorem 3 and Corollary 5) and that the Morava K-theories are the prime fields of the stable homotopy category ([HS98], Proposition 1.8).

10. **Thick Subcategories and Periodicity (23.7. - Fabian Lenhardt):** Formulate and prove the thick subcategory theorem (Theorem 7 of [HS98]). If time permits formulate and motivate the periodicity theorem ([Rav92], 1.5.4, or [HS98], Theorem 9) and indicate very briefly how to use the thick subcategory theorem in the proof. The periodicity theorem answers the second question posed above.

References

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[Böd87] C.-F. Bödigheimer, *Stable Splittings of Mapping Spaces.* Springer LNM 1286 (1987), pp. 174-187.

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[Hop08] M. Hopkins, *The Mathematical Work of Douglas C. Ravenel.* Homology, Homotopy and Applications, vol. 10(3), 2008, pp. 1-13. Online at http://intlpress.com/HHA/v10/n3/a1/v10n3a1.pdf

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[Rav92] D.C. Ravenel, *Nilpotence and Periodicity in Stable Homotopy Theory.* Princeton University Press, 1992.

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