Factorability, Discrete Morse Theory and a Reformulation of $K(\pi,1)$-conjecture

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1 Introduction

This thesis investigates some combinatorial properties of different groups and monoids with a view towards their classifying spaces and their homology. One of the methods to compute the homology of a group is to consider the bar complex of this group. Yet, the bar complex is very large and hard to deal with. While studying the homology of moduli spaces, C.-F. Bödigheimer and B. Visy discovered that symmetric groups have an additional structure allowing to reduce the bar complex to a considerably smaller one. This structure is a particular choice of a transposition for each permutation, which can be split off to obtain a new permutation. The new permutation in turn can be written as a shorter product of transpositions. This procedure gives us a preferred way of writing each permutation as a product of transpositions, and these choices are coherent in some precise sense. C.-F. Bödigheimer and B. Visy abstracted this structure and defined the notion of a factorable group (cf. [60]).

A factorability structure depends not only on the group or monoid itself but also on the chosen generating system. The factorability structure consists of a factorization map which assigns to each element of the monoid a preferred generator that is split off in a geodesic, i.e., word-length-preserving way. This map is subject to several axioms, which in particular ensure a certain, non-obvious compatibility with the multiplication in the monoid. Such a factorability structure yields a choice of geodesic normal forms, i.e., minimal representatives of each element of the monoid in terms of the chosen generating system.

Later on, R. Wang and A. Heß extended the definition of factorability to categories and monoids in [61] and [42]. The definitions are set up so that there exists a quite small complex computing the homology of those objects. This was partially shown by B. Visy and R. Wang and in general by A. Heß in his Ph.D. thesis ([42]) using discrete Morse theory.

The first aim of this thesis is to find groups and monoids equipped with such a factorability structure. Among the few families of examples besides symmetric groups, the dihedral groups and certain Thompson groups and monoids were known to carry an interesting factorability structure. One should keep in mind that the existence of a factorability structure depends on the choice of the generating system. For example, the symmetric groups are factorable with all transpositions as a generating system, but they do not admit a factorability structure if we take the generating system of simple transpositions. One obstruction to being a factorable monoid is the theorem due to M. Rodenhausen stating that the monoid has to admit for the given generating system a presentation with relations of length at most 4. Since there is no published version of this theorem till now, we include a proof into this thesis.

Here, we give several further examples. One of the main results provides factorability structures on the wide class of left locally Gaussian monoids (see Theorem 4.2.5):

Theorem. *Let M be a left locally Gaussian monoid, $\mathcal{E}$ a generating subset of M that is closed under left least common multiple and left complement. Then $(M, \mathcal{E})$ is factorable.*

This provides a large new class of factorable monoids. This class includes in particular the Artin monoids introduced by E. Brieskorn and K. Saito ([10]) to study the corresponding Artin groups. Locally left Gaussian monoids were introduced by P. Dehornoy and Y. Lafont in [28]. Similar concepts have already been defined by P. Dehornoy in [24] and by P. Dehornoy and L. Paris in [29]. These concepts were developed to abstract and generalize the work by F. Garside ([38]), where he solves the word problem and the conjugation problem for braid groups. There are several further treatments of these and similar structures in the literature, mostly united by the term "Garside theory", e.g. in [27], [39], [32]. For a certain subclass of left locally Gaussian monoids, the Garside monoids, it is possible to extend the factorability structure to the group of fractions of the monoid. In particular, one obtains a factorability structure on the braid groups and, more generally, on all Artin groups of finite type.

This leads us to a further important question of this thesis: What can be said about homology of Artin groups? Artin groups are closely related to Coxeter groups. Coxeter groups can be seen as a generalization of the finite subgroups of $O(n)$ generated by reflections. Like these, Coxeter groups have standard real representations by isometries with respect to some associated symmetric bilinear form. For Artin groups of finite type, i.e., for those Artin groups which have a positive definite associated bilinear form, there are well-known small models for the classifying spaces (as proved first by P. Deligne in [30]) given by certain hyperplane complements. The definition of such hyperplane complements makes perfect sense for all Artin groups. They are known to have always the corresponding Artin group as fundamental group. Thus, it is natural to ask whether those hyperplane complements are also classifying spaces for general Artin groups. This is exactly the subject of the still open $K(\pi, 1)$-conjecture. It is sometimes attributed to V. Arnold, R. Thom and F. Pham.

Conjecture ($K(\pi, 1)$-conjecture). *Let G be an Artin group and W the corresponding Coxeter group. Let V be the standard representation of W by isometries, and denote by $R \subset W$ the set of elements r mapped to reflections in hyperplanes $H_r \subset V$. Then the quotient of the complex hyperplane complement $V \otimes \mathbb{C} \setminus \bigcup_{r \in R} H_r \otimes \mathbb{C}$ by the W-action is a $K(G, 1)$-space.*

This conjecture would in particular imply the existence of finite-dimensional manifold models for $K(G, 1)$-spaces for Artin groups G. This conjecture is not solved in general, but known to be true in many cases. There are many good expositions on this topic, e.g. [16], [34], [17], [18], [39], [54]. They give also several equivalent formulations of the $K(\pi, 1)$-conjecture. The one which will be important for us is the following (cf. e.g. [18]):

Conjecture ($K(\pi, 1)$-conjecture II). *Let G be an Artin group and let $\mathcal{C}$ be the following category: The objects are the subsets I of the Coxeter generating set S such that the Coxeter subgroup $W(I)$ of W is finite. The morphisms are given by*

$$\mathcal{C}(I, J) = \begin{cases} G(J), & \text{if } I \subset J \\ \varnothing, & \text{else.} \end{cases}$$

Here, we denote by $W(I) \subset W$ and $G(J) \subset G$ the subgroups generated by I and J, respectively. The composition of $g \in \mathcal{C}(J, K) = G(K)$ and $h \in \mathcal{C}(I, J) = G(J)$ is given by $g\iota_{JK}(h)$, where ι_{JK} is the inclusion of $G(J)$ into $G(K)$. Then BG is homotopy equivalent to $B\mathcal{C}$.

Loosely speaking, we can rephrase this formulation of the $K(\pi, 1)$-conjecture by saying that BG is already completely described by the subgroups of G of finite type and their relationship. We will repeat this formulation combined with the language of Grothendieck constructions as Conjecture 5.3.9.

Besides the finite type Artin groups, also other Artin groups are known to satisfy the $K(\pi, 1)$-conjecture, e.g. Artin groups of FC-type ([18]) and Artin groups of large type ([40]). A more detailed account of the state of the conjecture can be found e.g. in [34] and [16]. In this thesis, we are going to give an alternative proof to a recent theorem first proved by N. Dobrinskaya which reformulates the $K(\pi, 1)$-conjecture. It connects the $K(\pi, 1)$-conjecture with the classifying space of the corresponding Artin monoid. We will reprove the following theorem in Section 5.3.

Theorem (cf. [33]). *The $K(\pi, 1)$-conjecture holds for G if and only if the inclusion $BM \to BG$ is a homotopy equivalence, where M is the Artin monoid associated to the Artin group G.*

Our proof is based on a completely different method than the original proof: Our main tool will be discrete Morse theory, a combinatorial analogue of the usual Morse theory, developed for well-behaved CW-complexes by K. Brown ([12]) and R. Forman ([36]). For technical reasons, we will need the slightly more general version due to E. Batzies ([4]).

Back to factorability, we will exhibit two new families of factorable groups: The orthogonal groups $O(V)$ (found jointly with C.-F. Bödigheimer) and the Coxeter groups of the B-series, both with all reflections as a generating system. Whether the D-series and the exceptional finite Coxeter groups are factorable with respect to all reflections, is still open.

For the more systematic treatment of factorable monoids, we deal with the question whether the short-relation presentation of a factorable monoid gives rise to a complete rewriting system. A rewriting system is just a monoid presentation with a preferred choice of the direction in which each relation may be applied. Completeness guarantees in particular that the rewriting rules lead to unique normal forms. Moreover, a complete rewriting system on a monoid provides a better understanding of its homology.

The question about the connection between factorability and complete rewriting systems was posed by A. Heß. It naturally arises since a factorability structure quite obviously yields a rewriting system, and this rewriting system induces a matching (a discrete analogue of the gradient vector field in Morse theory), which is exactly the one described by A. Heß in his thesis ([42]). Although this matching is always noetherian, this turns out not to be the case for the rewriting systems coming from factorability structures. Indeed, counterexamples exist even if this rewriting system is finite and the

monoid described by it is right-cancellative. An example of such a monoid is given in Section 7.1. Yet, there are several cases where the corresponding rewriting system is noetherian, as for example in the following theorem (see 7.4.5):

Theorem. *The rewriting system associated to the factorability structure on a left locally Gaussian monoid as above is complete.*

This statement generalizes the complete rewriting systems for Garside groups described in [41].

Now we describe in more detail the plan of the thesis. In Chapter 2, we report on basic definitions and existing results about factorability, including the proof of an alternative description by M. Rodenhausen.

In Chapter 3, we give a brief overview of rewriting systems and discrete Morse theory in two different flavors, and of their connection. We also include some auxiliary results we are going to use later. This section is quite technical and may be skipped during first reading and consulted for definitions as needed.

In Chapter 4, we first report on basic Garside theory. Then we explore the connection between factorability and Garside theory. Furthermore, we prove a generalization of the theorem by R. Charney, J. Meier and K. Whittlesey ([20]) providing a small chain complex for the homology of a class of left locally Gaussian monoids. This proof is joint work with A. Heß. We will also show that Thompson monoids, closely connected to Thompson's group F and introduced by A. Heß([42]), are left locally Gausssian. Furthermore, we will describe a new factorability structure induced by this fact.

In Chapter 5, we introduce several small complexes computing the homology of Artin monoids. Furthermore, we reprove the theorem of N. Dobrinskaya ([33]) by means of discrete Morse theory in Section 5.3.

In Chapter 6, we report on work of T. Brady and C. Watt ([9]), which allows us to describe a factorability structure on the orthogonal group. This structure is inherited by the Coxeter groups of the B-series. It is also compatible with the Visy factorability structure on the symmetric groups, which are exactly the Coxeter groups of the A-series. In Section 6.3, the factorability structure on the orthogonal groups is shown to coincide with the geometric description proposed by C.-F. Bödigheimer.

In Chapter 7, we deal with the connection between factorability structures and rewriting systems: We show that a factorability structure always yields a rewriting system, which is in general not necessarily noetherian, and exhibit some cases where noetherianity yet holds.

Acknowledgements

2 Factorability Structures

In this chapter, we are going to define factorability structures. We will collect some basic facts and notation. In Section 2.3, we are going to describe a criterion for factorability due to M. Rodenhausen.

2.1 Factorability: Basic Definitions

Factorability structures are a central object of study in this thesis. The idea is to provide a special sort of structure on groups (and later on monoids) which allows to get, starting with the bar complex, a much smaller complex for computing group homology (or corresponding analogues). The definition of this structure is due to B. Visy ([60]) and C.-F. Bödigheimer. It was generalized to monoids by R. Wang([61]) and A. Heß ([42]).

The first object we want to define is a normed group.

Definition 2.1.1. *Let G be a group. A map $N\colon G \to \mathbb{N}_0$ is called a **norm** on G if the following conditions hold:*

1. *The equality $N(g) = 0$ holds if and only if $g = 1$.*

2. *For any $g \in G$, we have $N(g) = N(g^{-1})$.*

3. *The triangle inequality holds: For any $g, h \in G$, we have $N(gh) \leq N(g) + N(h)$.*

Natural examples for norms are provided by the word length with respect to some chosen generating system. It turns out that all norms we are going to deal with are word lengths with respect to some generating system.

Now, for any group G with a norm N we can define the notion of factorability, as originally done by B. Visy and C.-F. Bödigheimer:

Definition 2.1.2 ([60]). *Let (G, N) be a normed group with multiplication μ. Let $T(G)$ be the set of elements of minimal positive norm. A **factorization map** is a function $\eta = (\overline{\eta}, \eta')\colon G \to G \times G$ with the following properties:*

(F1) For all $g \in G$, we have $g = \overline{\eta}(g)\eta'(g)$.

(F2) For all $g \in G$, we have $N(g) = N(\overline{\eta}(g)) + N(\eta'(g))$.

(F3) For all $g \in G \setminus \{1\}$, the element $\eta'(g)$ lies in $T(G)$.

*The group (G, N) is called **factorable** if it admits a factorization map η with the following additional properties: Define $\alpha = \eta \circ \mu$ and $\beta = (\mu \times \mathrm{id})(\mathrm{id} \times \eta)(\mathrm{id} \times \mu)(\eta \times \mathrm{id})$. Then η must satisfy for all $a \in G, b \in T(G)$:*

(F4) $N(\alpha(a, b)) = N(a) + N(b) \Leftrightarrow N(\beta(a, b)) = N(a) + N(b)$.

(F5) $N(\alpha(a, b)) = N(a) + N(b)$ implies $\alpha(a, b) = \beta(a, b)$.

We will often denote $\eta'(g)$ by g' and $\overline{\eta}(g)$ by $\overline{g}$. We call $\overline{g}$ the **remainder** of g and g' the **prefix** of g.

Remark 2.1.3. *The properties (F4) and (F5) guarantee to a certain extent the compatibility of the factorization map with multiplication. We can depict the compositions α and β in the following diagram (note that in general, it does not commute properly; it has to commute only in the "graded sense", i.e. either both compositions lower the norm or both compositions preserve the norm. In the latter case, we ask for the commutativity of the diagram.)*

$$
\begin{array}{ccc}
G \times T & \xrightarrow{\eta \times \mathrm{id}} & G \times \widetilde{T} \times T \\
\downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mathrm{id}\,\times\mu} \\
 & G \times G \xrightarrow{\mathrm{id}\,\times\eta} & G \times G \times \widetilde{T} \\
 & & \downarrow{\scriptstyle \mu\times\mathrm{id}} \\
G & \xrightarrow{\hspace{2cm}\eta\hspace{2cm}} & G \times \widetilde{T}
\end{array}
$$

where $\widetilde{T} = T \cup \{1\}$. Written once again for single elements, we have

$$
\begin{array}{ccc}
(g,t) \vdash\!-\!\rightarrow (\overline{g}, g', t) \\
\quad\downarrow & & \downarrow \\
 & (\overline{g}, g't) \vdash\!-\!-\!\rightarrow (\overline{g}, \overline{g't}, (g't)') \\
 & & \downarrow \\
gt \vdash\!-\!-\!\rightarrow (\overline{gt}, (gt)') \quad & (\overline{g}\cdot\overline{g't}, (g't)')
\end{array}
$$

Note that in the original Definition 3.1.3 of [60], the conditions (F4) and (F5) are formulated for pairs of group elements and not only for $G \times T$. Yet, these definitions are shown to be equivalent in Proposition 3.1.8 of [60].

Definition 2.1.4. *In a normed group (G, N), a pair (g, h) of group elements is called a **geodesic pair** if $N(gh) = N(g) + N(h)$.*

*In a monoid M with a chosen generating set $\mathcal{E}$, we call a pair $a, b \in M$ **geodesic** if $N_{\mathcal{E}}(ab) = N_{\mathcal{E}}(a) + N_{\mathcal{E}}(b)$, where $N_{\mathcal{E}}$ denotes the word length with respect to $\mathcal{E}$.*

Remark 2.1.5. *By a result of Visy([60], Corollary 3.1.7), given a factorization map η on a normed group (G, N) and a geodesic pair (g, h), the pairs (g', h) and $(\overline{g}, \overline{g'h})$ are automatically geodesic. Moreover, if we use word-by-word the same definition of factorization map for a monoid, the same property will hold.*

There is an alternative description of the last two axioms, which is often easier to handle.

Lemma 2.1.6 ([60]). *For a factorization map, the conditions (F4) and (F5) for a pair $a \in G$ and $b \in T(G)$ are equivalent to: If both $(\eta'(a), b)$ and $(\bar{\eta}(a), \bar{\eta}(\eta'(a)b))$ are geodesic pairs, then (a, b) is a geodesic pair and the equality $\eta'(ab) = \eta'(\eta'(a)b)$ holds. For monoids, we need to add the condition that if both pairs are geodesic, both $\eta'(ab) = \eta'(\eta'(a)b)$ and $\bar{\eta}(a)\bar{\eta}(\eta'(a)b) = \bar{\eta}(ab)$ hold. In groups or, more generally, right-cancellative monoids, this last condition holds automatically by cancellation.*

Note furthermore that if $\eta'(ab) = \eta'(\eta'(a)b)$ and $\bar{\eta}(a)\bar{\eta}(\eta'(a)b) = \bar{\eta}(ab)$ hold for some pair (a, b), the condition (F4) for this pair is automatically satisfied.

The following (folklore) observation allows us to attribute a factorability structure to a group with a chosen generating system instead of a normed group.

Lemma 2.1.7. *Let (G, N, η) be a factorable group and let $T = T(G)$ be the set of elements of minimal positive norm m. Then for any element $g \in G$, we have $N(g) = m \cdot N_T(g)$, where N_T denotes the word length norm with respect to T.*

Proof. First, we observe that for any norm N with $N(x) = m$ for all $x \in T$, we have $N(g) \leq m \cdot N_T(g)$ for any $g \in G$. To see this, let $N_T(g) = k$, then there are $g_1, \ldots, g_k \in T$ such that $g = g_k \ldots g_1$. By iterated use of the triangle inequality, we obtain

$$N(g) \leq N(g_k) + N(g_{k-1}) + \ldots + N(g_1) = m \cdot k = m \cdot N_T(g).$$

Now we are going to prove the other inequality for the case of factorable groups. The claim is certainly true for $g = 1$ and for $g \in T$. We proceed by induction on $N(g)$. Assume $N(g) = k > m$, and the claim is already shown for elements of smaller norm. Thus we already know that $N(\bar{g}) = mN_T(\bar{g})$ and $N(g') = m$. By the triangle inequality, $m \cdot N_T(g) \leq m \cdot N_T(\bar{g}) + m = N(g)$. This completes the proof. $\qquad \square$

From now on, we are going to consider a group with a chosen generating system when looking for factorability structures; the norm considered is then the corresponding word length, i.e., we set $m = 1$ since any other norm is just a multiple of this one. We require the generating system to be closed under taking inverses. Note that the choice of the generating system does matter. We will demonstrate it first by some examples; more advanced examples come up in later chapters of this thesis.

Example 2.1.8.   1. *Any group is factorable if we choose $T = G \setminus \{1\}$ and set $\eta(g) = (1, g)$ for all $g \in G$. This obviously satisfies the axioms of Definition 2.1.2.*

  2. *The free group on $\{x_1, \ldots, x_n\}$ is factorable with respect to the generating system $T = \{x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}\}$. Here, one can write each element x uniquely in the form $x_{i_k}^{\varepsilon_k} \ldots x_{i_1}^{\varepsilon_1}$ with $1 \leq i_j \leq n$ and $\varepsilon_j \in \{\pm 1\}$ if we require k to be minimal. For such an x, define*

$$\eta(x) = (x_{i_k}^{\varepsilon_k} \ldots x_{i_2}^{\varepsilon_2}, x_{i_1}^{\varepsilon_1}).$$

This defines a factorability structure (cf. [60]).

3. *The symmetric group $\mathfrak{S}_n$ with the generating system of all transpositions is the most important example given in the thesis of B. Visy ([60]). He defines η as follows. For any permutation $\sigma \in \mathfrak{S}_n$, let $k \in \{1, \ldots, n\}$ be the maximal number not fixed by σ. Then we set $\eta'(\sigma)$ to be the transposition $(k, \sigma^{-1}(k))$ and $\bar{\sigma} = \sigma \cdot (\eta'(\sigma))^{-1}$. It can be checked to define a factorability structure on $\mathfrak{S}_n$.*

There is a generalization of factorability to arbitrary monoids due to A. Heß. Note that the Definition 2.1.2 does not use inverses so it could be directly transferred to monoids. This property will be later called **weakly factorable**. Yet, it turns out in general not to be the right definition for our purposes, in particular, in order to obtain a small chain complex computing the monoid homology. A. Heß introduces instead the following definition.

Definition 2.1.9 (Heß[42])**.** *Let M be a monoid and $\mathcal{E}$ a generating set. Let*

$$\eta \colon M \to M \times M$$

*be a factorization map, i.e., let it satisfy the analogs of (F1)-(F3) of Definition 2.1.2. For $1 \leq i \leq n-1$, we denote by f_i the map $M^n \to M^n$ which assigns to $(x_n, \ldots, x_1) \in M^n$ the tuple $(x_n, \ldots, x_{i+2}, \eta(x_{i+1}x_i), x_{i-1}, \ldots, x_1)$. We define η to be a **factorability structure** if the three maps*

$$f_1f_2f_1f_2, \ f_2f_1f_2, \ f_2f_1f_2f_1 \colon M^3 \to M^3$$

*are equal in the graded sense, i.e., for each tuple $(x_3, x_2, x_1) \in M^3$, the three maps agree or each of them lowers the sum of the norms of the entries. In this case, we call the triple $(M, \mathcal{E}, \eta)$ a **factorable monoid**.*

Remark 2.1.10. *To illustrate the situation, we will depict f_i by the following diagram:*
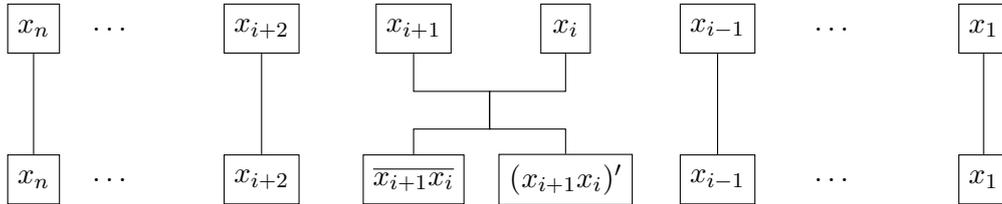
Notation 2.1.11. *Similarly to the notation of the last remark, if we have any map $\alpha \colon X^k \to X^l$ for some set X and some natural numbers k and l, we will define maps $\alpha_i \colon X^n \to X^{n-k+l}$ for $n \geq k$ and $1 \leq i \leq n-k+1$ via*

$$\alpha_i(x_n, \ldots, x_1) = (x_n, \ldots, x_{i+k}, \alpha(x_{i+k-1}, \ldots, x_i), x_{i-1}, \ldots, x_1).$$

Since the Definition 2.1.9 above is rather hard to check, we are going to use also an equivalent description given by A. Heß. We will now define the recognition principle and then formulate this equivalent description.

Definition 2.1.12 ([42])**.** *Let M be a monoid, let $\mathcal{E}$ be a generating system for M and let $\eta\colon M \to M \times M$ be a factorization map, i.e., it satisfies the analogs of (F1)-(F3) of Definition 2.1.2. We say that η satisfies the **recognition principle** if for all $m \in M$, $a \in \mathcal{E}$, the pair (m, a) satisfies $\eta(ma) = (m, a)$ if and only if the pair (m', a) satisfies $\eta(m'a) = (m', a)$, where $m' = \eta'(m)$.*

Theorem 2.1.13 (Heß[42], Theorem 2.2.6)**.** *Let M be a monoid, let $\mathcal{E}$ be a generating system for M and let $\eta\colon M \to M \times M$ be a factorization map. Then η is a factorability structure on M in the sense of Definition 2.1.9 if and only if it satisfies the conditions (F4) and (F5) of the Definition 2.1.2 and in addition the recognition principle 2.1.12.*

The following corollary will be important in our setting.

Corollary 2.1.14 (Heß [42], Sections 2.1-2.2)**.** *For right cancellative monoids, the notion of factorability as in Definition 2.1.9 coincides with the one given by the monoid version of Definition 2.1.2.*

An important property of factorable monoids is the existence of well-behaved normal forms. Indeed, if we start with an element $x \in M$, we first write it as $x = \overline{x}x'$. We can continue with $\overline{x}$ and write it again as $\overline{x} = \overline{\overline{x}} \cdot (\overline{x})'$. Inductively, we can write x as a product $x_k \dots x_1$ of $k = N(x)$ generators. This normal form has the property of being everywhere stable. We start by the definition of this notion.

Definition 2.1.15 (Heß [42], Rodenhausen [53])**.** *Let $(M, \mathcal{E}, \eta)$ be a factorable monoid. We say a tuple $(x_n, \dots, x_1) \in M^n$ is **stable** at the i-th position if $\eta(x_{i+1}x_i) = (x_{i+1}, x_i)$. We call a tuple **everywhere stable** if it is stable at each position.*

The tuple $(x_k, \dots, x_1)$ associated to $x \in M$ by the procedure defined above has now the following property:

Lemma 2.1.16 (Heß [42] Remark 2.1.27, Rodenhausen [53])**.** *Let $(M, \mathcal{E}, \eta)$ be a factorable monoid. Then the tuple $(x_k, \dots, x_1)$ associated to $x \in M$ by the procedure as above is everywhere stable. This tuple associated to x will be called the **normal form** of x (with respect to this factorability structure).*

There is a special case of factorability which is particularly nice since it can simplify the differentials in the Visy complex, which will be defined later (cf. [42], Section 4.2).

Definition 2.1.17 (Heß, Rodenhausen)**.** *Let $(M, \mathcal{E}, \eta)$ be a factorable monoid. We call the factorability structure **braided factorable** if $f_1 f_2 f_1$ and $f_2 f_1 f_2$ are equal on $\mathcal{E}^3$ in the graded sense, i.e., for all $(a, b, c) \in \mathcal{E}^3$, the elements $f_1 f_2 f_1(a, b, c)$ and $f_2 f_1 f_2(a, b, c)$ have both norm less than 3 or are equal.*

2.2 Homology of Factorable Monoids

Now, we are going to describe a complex computing the homology of factorable groups and monoids which is smaller than the bar complex. Recall that the inhomogeneous

normalized bar complex $(\overline{B}_*M, d)$ associated to a monoid M is a chain complex with free $\mathbb{Z}$-modules $\overline{B}_nM$ generated by n-tuples of elements in M, written $[x_n|\ldots|x_1]$, with $x_i \neq 1$ for all $1 \leq i \leq n$. The differential d is given by the sum $\sum_{i=0}^{n}(-1)^i d_i$, where the d_i are defined on the basis by

$$d_i([x_n|\ldots|x_1]) = \begin{cases} [x_n|\ldots|x_2], & \text{if } i = 0, \\ [x_n|\ldots|x_{i+2}|x_{i+1}x_i|x_{i-1}|\ldots|x_1], & \text{if } 0 < i < n, \\ [x_{n-1}|\ldots|x_1], & \text{if } i = n. \end{cases}$$

A smaller complex for factorable groups was first introduced by B. Visy ([60]); it was shown to compute the homology of a factorable right-cancellative monoids with finite generating system by R. Wang ([61]). This statement was reproved by means of discrete Morse theory and extended to the case of general monoids by A. Heß ([42]). Furthermore, A. Heß showed that this complex comes from a free resolution of $\mathbb{Z}$ over $\mathbb{Z}M$. In wide parts of this section, we follow the exposition of [42].

The differentials of the complex are rather complicated. Here, we will not give the original definition but rather a description found by A. Heß. Before this, we need some further definitions. We will also need them later to make a link between factorability structures and rewriting systems.

First, we consider the monoids P_n, Q_n and Q'_n whose actions describe to a certain extent the effect of the factorability structure on the bar complex. We will need Q'_n in Chapter 7 to show in some cases that the rewriting system associated to a factorability structure is noetherian.

Definition 2.2.1 (Heß[42])**.** *Let F_n be a free monoid on letters $1, 2, \ldots, n$ (with the empty string as a neutral element). The elements of F_n will be denoted either like $(1\,2\,3\,4)$ or like $(1, 2, 3, 4)$ to increase the readability; sometimes, we also omit the brackets. Let $\sim_P$ be the congruence generated by*

$$ab \sim_P ba \text{ for } |a - b| \geq 2 \text{ and}$$
$$a^2 \sim_P a \text{ for } 1 \leq a \leq n.$$

Recall that a congruence is a left and right invariant equivalence relation. Let P_n be the quotient of F_n by this congruence. Define now a congruence $\sim_Q$ on P_n generated by

$$(k\ \ k+1\ \ k\ \ k+1) \sim_Q (k+1\ \ k\ \ k+1) \text{ and}$$
$$(k+1\ \ k\ \ k+1\ \ k) \sim_Q (k+1\ \ k\ \ k+1).$$

Let Q_n be the quotient of P_n by this congruence. Last, define a congruence $\sim$ on Q_n generated by the following relation: If for $I, J \in F_n$ the relation $kIJ \sim_Q IJ$ holds, $kI \sim_P Ik$ and k does not occur in I, then we set $kJ \sim J$. Define the quotient monoid of this congruence to be Q'_n.

The following evaluation lemma should motivate the definition of Q_n.

Lemma 2.2.2 (Heß[42], Section 2.2, Evaluation Lemma)**.** *Let $(M, \mathcal{E}, \eta)$ be a factorable monoid. For any sequence $I = (i_s, \ldots, i_1) \in F_n$, we define $f_I : M^{n+1} \to M^{n+1}$ to be the composition $f_{i_s} \circ f_{i_{s-1}} \circ \ldots \circ f_{i_1}$. Such a map descends to a map $f_I : \overline{B}_{n+1} M \to \overline{B}_{n+1} M$. If $I \sim_Q J$, then the maps*

$$f_I, f_J : \overline{B}_{n+1} M \to \overline{B}_{n+1} M$$

are equal in the graded sense.

In the "graded sense" means here that the following two conditions are satisfied. First, if $f_I([x_{n+1}| \ldots |x_1]) = [y_{n+1}| \ldots |y_1]$ and $f_J([x_{n+1}| \ldots |x_1]) = [z_{n+1}| \ldots |z_1]$, then

$$N_{\mathcal{E}}(x_{n+1}) + \ldots + N_{\mathcal{E}}(x_1) = N_{\mathcal{E}}(y_{n+1}) + \ldots + N_{\mathcal{E}}(y_1)$$

if and only if

$$N_{\mathcal{E}}(x_{n+1}) + \ldots + N_{\mathcal{E}}(x_1) = N_{\mathcal{E}}(z_{n+1}) + \ldots + N_{\mathcal{E}}(z_1)$$

holds, and in this case, $f_I([x_{n+1}| \ldots |x_1]) = f_J([x_{n+1}| \ldots |x_1])$. Second, the equation $f_I([x_{n+1}| \ldots |x_1]) = 0$ is equivalent to $f_J([x_{n+1}| \ldots |x_1]) = 0$.

Thus, instead of proving graded identities for the f_i's, we may often prove the corresponding identities in Q_n. We will also show later that a similar, but weaker evaluation lemma holds for Q'_n.

For later use, we fix some notation for those monoids and collect some facts about them.

Notation 2.2.3. *We denote by $\mathrm{sh}_k : F_{n-k} \to F_n$ the shift homomorphism induced by $i \mapsto i + k$.*

For $I, J \in F_n$, we write $I \subset J$ if I is a (possibly disconnected) subsequence of J.
Denote by I_a^b the sequence $(a \; a+1 \ldots \; b-1 \; b)$.
Denote by D_k the sequence $I_k^k I_{k-1}^k \ldots I_2^k I_1^k$.

The elements D_n play a very special role in the monoids Q_n.

Theorem 2.2.4 (Heß[42], Section 2.3)**.** *The element represented by D_n in Q_n is an absorbing element, i.e., for any $I \in F_n$, we have $ID_n \sim_Q D_n \sim_Q D_n I$.*

In particular, the Evaluation Lemma implies that $f_{D_n}(x_n, \ldots, x_1)$ is everywhere stable if f_{D_n} does not drop the norm. Even more is true:

Proposition 2.2.5 (Heß[42], Section 2.3)**.** *Let $(M, \mathcal{E}, \eta)$ be a factorable monoid. Let m be an element of M and let $m_n \ldots m_1$ be a minimal word in $\mathcal{E}$ representing m. Then $f_{D_{n-1}}(m_n, \ldots, m_1)$ is the normal form of m.*

The following definitions introduce particularly nice representatives of elements in P_n. These are going to make it easier to track down applications of some f_I to tuples of monoid elements.

Definition 2.2.6 (Heß[42])**.** *A sequence* $(i_s, \ldots, i_1) \in F_n$ *is called* ***left-most*** *if for every* $s > t \geq 1$, *the following holds: if* $|i_{t+1} - i_t| \geq 2$, *then* $i_{t+1} > i_t$. *In other words,* i_t *exceeds* i_{t+1} *at most by 1. A sequence* $(i_s, \ldots, i_1)$ *is called* ***reduced*** *if it contains no subsequent equal entries.*

The following criterion provides an equivalent description of left-most sequences.

Lemma 2.2.7 (Heß[42], Section 2.3)**.** *A sequence* $I = (i_s, \ldots, i_1) \in F_n$ *is left-most if and only if for every connected subsequence* J *holds: If* $a < b$ *and* $(a, b) \subset J$, *then* $I_a^b \subset J$.

A. Heß showed that we can always find left-most, reduced representatives for elements in P_n:

Proposition 2.2.8 (Heß[42], Section 2.3)**.** *Every sequence* $I \in F_n$ *is* $\sim_P$-*equivalent to a left-most, reduced one.*

The following lemma shows that choosing such a representative preserves existence of certain subsequences:

Lemma 2.2.9. *Let* $J \in F_n$ *be a sequence with* $I_1^n \subset J$. *Let furthermore* J' *be a left-most, reduced sequence with* $J' \sim_P J$. *Then* $I_1^n \subset J'$ *holds.*

Proof. This follows directly as one observes that whenever $(i, i+1)$ is a (possibly disconnected) subsequence of some word in F_n, those entries can never be interchanged using relations in P_n. $\qquad\square$

This finishes the list of general properties of P_n and Q_n which will be used later. We need a last definition before we can describe the Visy complex.

Notation 2.2.10. *Let* $\square_n$ *be the two-sided ideal in* Q_n *generated by all elements*

$$[2\,1\,2], [3\,2\,3], \ldots, [n\,n-1\,n].$$

Here, we use the square brackets to stress that we mean the elements of Q_n *represented by the corresponding strings.*

Thus, $\square_n$ *consists exactly of all elements of* Q_n *which have a representative with a connected subsequence of the form* $(i+1\,i\,i+1)$ *for some* $1 \leq i \leq n-1$.

Now we are ready to describe the complex mentioned above.

Definition 2.2.11. *Let* $(M, \mathcal{E}, \eta)$ *be a factorable monoid. We are going to define a chain complex* $(\mathbb{V}, \partial^{\mathbb{V}})$ *over* $\mathbb{Z}$, *computing the homology of* M. *(By this, we mean the homology of the inhomogeneous bar complex of* M.) *We will call this complex* ***Visy complex***. *The modules* $\mathbb{V}_n$ *are free modules over* $\mathbb{Z}$ *with basis given by the* n-*tuples* $[x_n|\ldots|x_1]$ *with* $x_i \in \mathcal{E}$ *and* $\eta(x_{i+1}x_i) \neq (x_{i+1}, x_i)$ *for all* $1 \leq i \leq n-1$.

The differentials can be written as follows:

$$\partial^{\mathbb{V}} = \pi_{n-1} \circ d \circ \left(\sum_{\alpha \in Q_{n-1} \backslash \square_{n-1}} (-1)^{l(\alpha)} f_\alpha \right) \circ i_n.$$

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Here, i_n denotes the inclusion of $\mathbb{V}_n$ into the n-th module of the normalized inhomogeneous bar complex $\overline{B}_n(M)$, d denotes the n-th differential in the same bar complex, and $\pi_{n-1}\colon \overline{B}_n(M) \to \mathbb{V}_n$ denotes the projection. Last, by $l(\alpha)$ we denote the word length of α in Q_{n-1} with respect to the generating system $\{[1], [2], \ldots, [n-1]\} \subset Q_{n-1}$.

The property which originally motivated the definition of factorable groups may now be formulated as follows:

Theorem 2.2.12 (Heß [42], Visy [60], Wang[61])**.** *Let $(M, \mathcal{E}, \eta)$ be a factorable monoid. Then the homology of the Visy complex $(\mathbb{V}, \partial^{\mathbb{V}})$ is isomorphic to the monoid homology of M, i.e., to the homology of the normalized inhomogeneous bar complex $(\overline{B}_* M, d)$.*

2.3 An Alternative Description of Factorability

M.Rodenhausen gave the following alternative description of factorability, which needs to be defined only on pairs of generators. A byproduct of this description is the existence of very special presentations for factorable monoids. Since the proofs of M. Rodenhausen are unpublished till now, we will write them down with his kind permission.

Definition 2.3.1. *(Rodenhausen, [53]) Let M be a monoid and $\mathcal{E}$ a generating system of this monoid. Denote by $\mathcal{E}^+$ the union of this generating system with $\{1\}$, and by $\mathcal{E}^*$ the free monoid generated by $\mathcal{E}$. In this section, we always assume $1 \notin \mathcal{E}$. Then a **local factorability structure** is a map*

$$\varphi\colon \mathcal{E}^+ \times \mathcal{E}^+ \to \mathcal{E}^+ \times \mathcal{E}^+$$

with the following properties:

1. *$M \cong \langle \mathcal{E} | (a, b) = \varphi(a, b) \rangle$.*

2. *Idempotency: $\varphi^2 = \varphi$.*

3. *Value on norm 1 elements: $\varphi(a, 1) = (1, a)$.*

4. *Stability for triples: $\varphi_2 \varphi_1 \varphi_2(a, b, c)$ is (φ-)totally stable (i.e., applying any φ_i to this tuple leaves it unchanged) or contains a 1, for all $a, b, c \in \mathcal{E}$.*

5. *Normal form condition: $\mathrm{NF}(a, b, c) = \mathrm{NF}(\varphi_1(a, b, c))$ for all $a, b, c \in \mathcal{E}$.*

(Recall we use the notational convention 2.1.11 to define φ_i). Here, the normal form of a tuple $(a_n, \ldots, a_1)$ is an element of $\mathcal{E}^*$ defined inductively as follows: The normal form of a string containing 1 is the normal form of the same string with 1 removed. For a string not containing 1, define

$$\mathrm{NF}(a_n, \ldots, a_1) = \begin{cases} \varphi_{n-1} \ldots \varphi_1(\mathrm{NF}(a_n, \ldots, a_2), a_1), & \text{if it contains no } 1; \\ \mathrm{NF}(\varphi_{n-1} \ldots \varphi_1(\mathrm{NF}(a_n, \ldots, a_2), a_1)), & \text{otherwise.} \end{cases}$$

*Define $\mathrm{NF}(a) = a$ for all $a \in \mathcal{E}$ and $\mathrm{NF}(()) = ()$. Furthermore, we call a string of the form $(1, 1, \ldots, 1, \mathrm{NF}(x))$ an **extended** normal form of x.*

Theorem 2.3.2. *(Rodenhausen, [53]) If a monoid M with a generating system $\mathcal{E}$ is factorable, then $\varphi(a,b) = \eta(ab)$ defines a local factorability structure on this monoid. Conversely, one can construct out of a local factorability structure a factorability structure in the usual sense, and those share the same normal forms. These constructions are inverse to each other.*

The proof of this theorem will be subdivided in several smaller steps. First, we are going to indicate the connection between being a normal form and total stability.

Proposition 2.3.3. *([53]) Let $\mathcal{E}$ be a set. Let*

$$\varphi \colon \mathcal{E}^+ \times \mathcal{E}^+ \to \mathcal{E}^+ \times \mathcal{E}^+$$

be a map with $\varphi^2 = \varphi$ and $\varphi(a,1) = (1,a)$. (In other words, we start with any set $\mathcal{E}$ and a map φ satisfying the second and third conditions of Definition 2.3.1.) Furthermore, we define the normal form function $\mathrm{NF} \colon \mathcal{E}^* \to \mathcal{E}^*$ as in Definition 2.3.1. Then if a word $(a_n, \ldots, a_1)$ with $a_i \in \mathcal{E}$ is totally stable, i.e., if $\varphi_j(a_n, \ldots, a_1) = (a_n, \ldots, a_1)$ for all $1 \leq j \leq n-1$, then it is its own normal form.

If in addition the stability for triples condition of Definition 2.3.1 is satisfied by φ, then also the converse holds: Any normal form is totally stable.

Proof. We will first show that any totally stable word (without 1's) is its own normal form. We will prove the statement by induction on the length of the word n. For $n = 0$ or $n = 1$, $\mathrm{NF}(()) = ()$ and $\mathrm{NF}(a) = a$ for all $a \in \mathcal{E}$. These are by definition all totally stable and in their normal form. For a stable pair (a_2, a_1), we have to apply φ, which does not change the pair by assumption. In particular, no 1's occur, so (a_2, a_1) is its own normal form.

Now assume we have already shown the claim for all natural numbers smaller than n. We are now going to prove the claim for strings of length n. Observe that $(a_n, \ldots, a_2)$ is also totally stable and of length $n-1$, so it is its own normal form. For computing the normal form of $(a_n, \ldots, a_1)$, we now have to compute $\varphi_{n-1} \ldots \varphi_1(a_n, \ldots, a_1)$, which is by assumption just $(a_n, \ldots, a_1)$ itself. Thus, in particular, it does not contain 1's and this is the normal form itself. This yields the first claim.

Now assume the stability for triples condition holds for φ. We want to show that any normal form is totally stable. We proceed again by induction on the length n of the string. The statement is clear for $n = 0, 1, 2$. Now assume we have already shown the statement for all strings of length less than n. We want to show that $\mathrm{NF}(a_n, \ldots, a_1)$ is totally stable. We may assume that $\varphi_{n-1} \ldots \varphi_1(\mathrm{NF}(a_n, \ldots, a_2), a_1)$ is the normal form of $(a_n, \ldots, a_1)$ and in particular does not contain 1's. Indeed, otherwise $\mathrm{NF}(a_n, \ldots, a_1)$ is at the same time the normal form of a shorter string, thus, by the induction hypothesis, totally stable. Hence, applying φ_{n-1} to $\mathrm{NF}(a_n, \ldots, a_1)$ does not change the tuple due to $\varphi_{n-1}^2 = \varphi_{n-1}$. Furthermore, since $\mathrm{NF}(a_n, \ldots, a_2)$ is totally stable by the induction hypothesis, we know that

$$\varphi_j(\mathrm{NF}(a_n, \ldots, a_2), a_1) = (\mathrm{NF}(a_n, \ldots, a_2), a_1)$$

for all $2 \le j \le n-1$.

Note that the application of $\varphi_{i+1}\varphi_i\varphi_{i+1}$ to $\varphi_{i-1}\varphi_{i-2}\dots\varphi_1(\mathrm{NF}(a_n,\dots,a_2),a_1)$ does not produce a 1: Otherwise, there would be a 1 in

$$
\begin{aligned}
&\varphi_{n-1}\dots\varphi_{i+2}\varphi_{i+1}\varphi_i\varphi_{i+1}\varphi_{i-1}\varphi_{i-2}\dots\varphi_1(\mathrm{NF}(a_n,\dots,a_2),a_1) \\
&= \varphi_{n-1}\dots\varphi_{i+1}\varphi_i\varphi_{i-1}\dots\varphi_1\varphi_{i+1}(\mathrm{NF}(a_n,\dots,a_2),a_1) \\
&= \varphi_{n-1}\dots\varphi_{i+1}\varphi_i\varphi_{i-1}\dots\varphi_1(\mathrm{NF}(a_n,\dots,a_2),a_1) \\
&= \mathrm{NF}(a_n,\dots,a_1),
\end{aligned}
$$

and thus we would obtain a contradiction. Here, we used that φ_l and φ_k obviously commute whenever $|l-k| \ge 2$.

Using the commutativity again, we conclude for $1 \le i \le n-2$:

$$
\begin{aligned}
\varphi_i(\mathrm{NF}(a_n,\dots,a_1)) &= \varphi_i\varphi_{n-1}\dots\varphi_1(\mathrm{NF}(a_n,\dots,a_2),a_1) \\
&= \varphi_i\varphi_{n-1}\dots\varphi_1\varphi_{i+1}(\mathrm{NF}(a_n,\dots,a_2),a_1) \\
&= \varphi_{n-1}\dots\varphi_{i+2}\varphi_i\varphi_{i+1}\varphi_i\varphi_{i+1}\varphi_{i-1}\varphi_{i-2}\dots\varphi_1(\mathrm{NF}(a_n,\dots,a_2),a_1).
\end{aligned}
$$

As we already know that the application of $\varphi_{i+1}\varphi_i\varphi_{i+1}$ to

$$
\varphi_{i-1}\varphi_{i-2}\dots\varphi_1(\mathrm{NF}(a_n,\dots,a_2),a_1)
$$

does not produce a 1, the stability for triples condition implies that $\varphi_{i+1}\varphi_i\varphi_{i+1}$ makes the triple in positions $i+2, i+1, i$ totally stable, and thus

$$
\begin{aligned}
&\varphi_i\varphi_{i+1}\varphi_i\varphi_{i+1}\varphi_{i-1}\varphi_{i-2}\dots\varphi_1(\mathrm{NF}(a_n,\dots,a_2),a_1) \\
&= \varphi_{i+1}\varphi_i\varphi_{i+1}\varphi_{i-1}\varphi_{i-2}\dots\varphi_1(\mathrm{NF}(a_n,\dots,a_2),a_1).
\end{aligned}
$$

Performing the same manipulations backwards, we see that this implies the stability in position i. This yields the second claim. $\qquad\square$

This proposition immediately implies the following corollary.

Corollary 2.3.4. *Let $\mathcal{E}$ be a set and $\varphi\colon \mathcal{E}^+ \times \mathcal{E}^+ \to \mathcal{E}^+ \times \mathcal{E}^+$ a map satisfying the second, third and fourth condition of Definition 2.3.1. Let the normal form be defined as before. Then:*

1. *Building of normal form is idempotent:* $\mathrm{NF} \circ \mathrm{NF} = \mathrm{NF}$ *holds for all tuples.*

2. *Connected subwords of normal forms are normal forms.*

The following small observation about the normal form of triples is going to be helpful.

Lemma 2.3.5. *Let $\mathcal{E}$ be a set. Let*

$$
\varphi\colon \mathcal{E}^+ \times \mathcal{E}^+ \to \mathcal{E}^+ \times \mathcal{E}^+
$$

be a map with $\varphi^2 = \varphi$ and $\varphi(a,1) = (1,a)$ for which stability of triples holds. (In other words, we start with any set $\mathcal{E}$ and a map φ satisfying the second, third and fourth conditions of Definition 2.3.1.) Furthermore, we define the normal form function $\mathrm{NF}\colon \mathcal{E}^* \to \mathcal{E}^*$ as in Definition 2.3.1. Then it follows that an application of $\varphi_1\varphi_2\varphi_1\varphi_2$ to a triple (a,b,c) yields an extended normal form for this triple.

Proof. By Corollary 2.3.4, we have

$$\mathrm{NF}(a_n, \ldots, a_1) = \mathrm{NF}(\mathrm{NF}(a_n, \ldots, a_2), a_1)$$

for every tuple $(a_n, \ldots, a_1)$ with $a_i \in \mathcal{E}^+$. Thus,

$$\mathrm{NF}(a, b, c) = \mathrm{NF}(\mathrm{NF}(a, b), c) = \mathrm{NF}(\varphi(a, b), c).$$

If $\varphi(a, b) = (1, d)$ for some $d \in \mathcal{E}^*$, then $\varphi_1\varphi_2$ yields already an extended normal form of this triple and it is totally stable by the Proposition 2.3.3. If $\varphi(a, b)$ does not contain a 1, then by definition $\mathrm{NF}(a, b, c) = \varphi_2\varphi_1\varphi_2(a, b, c)$ if this does not contain a 1. If it does contain a 1, it is on the very left, and the application of φ_1 brings the triple into an extended normal form. $\square$

In order to define a factorability structure in the usual sense on a monoid with a local factorability structure, we have to check that the normal form of a monoid element does not depend on the word representing it. This is our next aim.

Proposition 2.3.6 (Rodenhausen)**.** *Let φ be a local factorability structure on $\mathcal{E}$, and let* NF *be the associated normal form as in Definition 2.3.1. Then for any $1 \leq j \leq n-1$ and all $(a_n, \ldots, a_1)$ with entries in $\mathcal{E}^+$, the following equality holds:*

$$\mathrm{NF}(a_n, \ldots, a_1) = \mathrm{NF}(\varphi_j(a_n, \ldots, a_1)).$$

Proof. Recall that by Corollary 2.3.4, we have

$$\mathrm{NF}(a_n, \ldots, a_1) = \mathrm{NF}(\mathrm{NF}(a_n, \ldots, a_2), a_1)$$

for every tuple $(a_n, \ldots, a_1)$ with $a_i \in \mathcal{E}^+$. Iterating this, we obtain furthermore for each $2 \leq k \leq n$:

$$
\begin{aligned}
\mathrm{NF}(a_n, \ldots, a_1) &= \mathrm{NF}(\mathrm{NF}(a_n, \ldots, a_2), a_1) \\
&= \mathrm{NF}(\mathrm{NF}(\mathrm{NF}(a_n, \ldots, a_3), a_2), a_1) \\
&= \mathrm{NF}(\mathrm{NF}(a_n, \ldots, a_3), a_2, a_1) \\
&= \ldots \\
&= \mathrm{NF}(\mathrm{NF}(a_n, \ldots, a_k), a_{k-1}, \ldots, a_1).
\end{aligned}
$$

Observe furthermore that applying $\varphi_{n-1} \ldots \varphi_1$ to the tuple $(\mathrm{NF}(a_n, \ldots, a_2), a_1)$ produces a one or yields the normal form.

We proceed by induction on n. For $n = 1$, there is nothing to prove. For $n = 2$, note that $\varphi_1(a_2, a_1)$ yields an extended normal form of the tuple. Thus, the claim follows since $\varphi^2 = \varphi$.

Now assume we have proved the statement for all numbers smaller than n. First, if $j \geq 2$, then we are done by induction hypothesis, since the following holds:

$$
\begin{aligned}
\mathrm{NF} \circ \varphi_j(a_n, \ldots, a_1) &= \mathrm{NF}(\mathrm{NF} \circ \varphi_{j-1}(a_n, \ldots, a_2), a_1) = \mathrm{NF}(\mathrm{NF}(a_n, \ldots, a_2), a_1) \\
&= \mathrm{NF}(a_n, \ldots, a_1).
\end{aligned}
$$

So we have to deal with the case $j = 1$. By the considerations above, we may assume that $(a_n, \ldots, a_3)$ is already in the normal form. We want to show that the application of

$$\varphi_{n-1} \ldots \varphi_2 \varphi_1 \varphi_{n-1} \ldots \varphi_3 \varphi_2$$

does not change the normal form. Note that it either produces the normal form or a 1 at the leftmost place. We have already seen that the application of $\varphi_{n-1} \ldots \varphi_2$ does not change the normal form since all indices are ≥ 2. If this application produces a 1, we obtain a new tuple with less than n non-trivial letters and application of any φ_k to it does not change the normal form by induction hypothesis. Thus, if $\varphi_{n-1} \ldots \varphi_2$ produces a 1, the concatenation above does not change the normal form. If $\varphi_{n-1} \ldots \varphi_2$ does not produce a 1, it brings the entries $(a_n, \ldots, a_2)$ into their normal form. Note then after this, the concatenation $\varphi_{n-1} \ldots \varphi_2 \varphi_1$ either produces the normal form or a 1 at some point. In the first case, we are immediately done by Corollary 2.3.4. In the second case, the normal form of the original tuple is defined as the normal form of the new tuple with 1 deleted. So the concatenation $\varphi_{n-1} \ldots \varphi_2 \varphi_1 \varphi_{n-1} \ldots \varphi_3 \varphi_2$ does not change the normal form.

Thus, we have shown that starting with a tuple $(a_n, \ldots, a_1)$ with $(a_n, \ldots, a_3)$ already in the normal form, then

$$\mathrm{NF}\left(\varphi_{n-1} \ldots \varphi_2 \varphi_1 \varphi_{n-1} \ldots \varphi_3 \varphi_2 (a_n, \ldots, a_1)\right) = \mathrm{NF}(a_n, \ldots, a_1)$$

holds. Rearranging now the applications of φ on the left-hand side according to $\varphi_i \varphi_k = \varphi_k \varphi_i$ for $|i - k| \geq 2$, we obtain

$$\varphi_{n-1} \ldots \varphi_2 \varphi_1 \varphi_{n-1} \ldots \varphi_3 \varphi_2 = \varphi_{n-1} \ldots \varphi_2 \varphi_{n-1} \ldots \varphi_3 \varphi_1 \varphi_2,$$

where on the right-hand side, we have only indices greater than 1 after the application of $\varphi_1 \varphi_2$. As already shown, those do not change the normal form, thus we obtain

$$
\begin{aligned}
\mathrm{NF}(a_n, \ldots, a_1) &= \mathrm{NF}\left(\varphi_{n-1} \ldots \varphi_2 \varphi_1 \varphi_{n-1} \ldots \varphi_3 \varphi_2 (a_n, \ldots, a_1)\right) \\
&= \mathrm{NF}(\varphi_1 \varphi_2 (a_n, \ldots, a_1))
\end{aligned}
$$

for any tuple $(a_n, \ldots, a_1)$ with $(a_n, \ldots, a_3)$ in normal form. Again, since we have already shown $\mathrm{NF} \circ \varphi_j = \mathrm{NF}$ for $j \geq 2$ for n letters, we see that we can continue our computation to obtain

$$\mathrm{NF}(a_n, \ldots, a_1) = \mathrm{NF}(\varphi_1 \varphi_2 (a_n, \ldots, a_1)) = \mathrm{NF}(\varphi_2 \varphi_1 \varphi_2 (a_n, \ldots, a_1)).$$

By the definition of local factorability, $\varphi_2 \varphi_1 \varphi_2$ makes the right-most triple totally stable or produces a 1. If it makes it totally stable and the result does not contain a 1, this yields by definition the normal form of the last triple. If this application does produce a 1, this 1 will appear at the very left of this triple (thus, on the third place from the right of the whole tuple), and we can use $\varphi_{n-1} \ldots \varphi_3$ to move this 1 to the very left without changing the normal form of the tuple. From here, we are dealing with a tuple with

at most $n-1$ non-empty letters, thus, by induction hypothesis, φ_1 does not change its normal form. All in all, we conclude for the case where $\varphi_2\varphi_1\varphi_2$ produces a 1:

$$
\begin{aligned}
\mathrm{NF}(\varphi_2\varphi_1\varphi_2(a_n,\ldots,a_1)) &= \mathrm{NF}(\varphi_{n-1}\ldots\varphi_3\varphi_2\varphi_1\varphi_2(a_n,\ldots,a_1)) \\
&= \mathrm{NF}(\varphi_1\varphi_{n-1}\ldots\varphi_3\varphi_2\varphi_1\varphi_2(a_n,\ldots,a_1)) \\
&= \mathrm{NF}(\varphi_{n-1}\ldots\varphi_3\varphi_1\varphi_2\varphi_1\varphi_2(a_n,\ldots,a_1)) \\
&= \mathrm{NF}(\varphi_1\varphi_2\varphi_1\varphi_2(a_n,\ldots,a_1)) \\
&= \mathrm{NF}(a_n,a_{n-1},\ldots,a_5,a_4,\mathrm{NF}(a_3,a_2,a_1)).
\end{aligned}
$$

In the last step, we use Lemma 2.3.5.

In any case, we have

$$
\mathrm{NF}(a_n,\ldots,a_1) = \mathrm{NF}(\varphi_2\varphi_1\varphi_2(a_n,\ldots,a_1)) = \mathrm{NF}(a_n,a_{n-1},\ldots,a_5,a_4,\mathrm{NF}(a_3,a_2,a_1))
$$

for tuples $(a_n,\ldots,a_1)$ with $(a_n,\ldots,a_3)$ in normal form. Since φ_1 does not affect the latter property, the claim now follows directly from this last observation using the normal form condition of local factorability, namely,

$$
\mathrm{NF}(a_3,a_2,a_1) = \mathrm{NF}(\varphi_1(a_3,a_2,a_1)).
$$

This completes the induction step and so the whole proof. $\qquad\square$

The proposition implies the following corollary.

Corollary 2.3.7. *Let M be a monoid with a generating system $\mathcal{E}$ and a local factorability structure φ. Then any element $m \in M$ has a unique normal form, i.e., a unique word in $\mathcal{E}^*$ which represents m and which is obtained by picking any representative of m and applying the normal form procedure to it. In other words, if $m = a_n\ldots a_1 = b_k\ldots b_1$ with $a_i, b_i \in \mathcal{E}^+$, then*

$$
\mathrm{NF}(a_n,\ldots,a_1) = \mathrm{NF}(b_k,\ldots,b_1).
$$

Proof. The only thing to show is the independence of the representative of m we are starting with. This follows immediately from the last proposition since all the relations in M are by definition given by φ. $\qquad\square$

Now we are going to make some preparations in order to prove the other implication of Theorem 2.3.2, i.e., in order to show that usual factorability induces a local factorability structure on the monoid.

Lemma 2.3.8. *Let $(M,\mathcal{E},\eta)$ be a factorable monoid, and set $\varphi(a,b) = \eta(ab)$ for all $a,b \in \mathcal{E}^+$. Then for any $a,b,c \in \mathcal{E}$, the word $\varphi_2\varphi_1\varphi_2(a,b,c)$ is totally φ-stable or contains a 1 on the very left. Furthermore, $\varphi_1\varphi_2\varphi_1\varphi_2(a,b,c)$ is totally φ-stable, and it equals the triple $(\overline{\eta}(\overline{\eta}(abc)),\eta'(\overline{\eta}(abc)),\eta'(abc))$.*

Proof. In this proof, we mean again with "stable" φ-stable.

First, observe that $\varphi(a, b)$ always gives a stable pair by (F1). Furthermore, we observe using (F2) that φ gives an idempotent map with $\varphi(a, 1) = (1, a)$. Moreover, for $x, y \in \mathcal{E}^+$, we know that $N_{\mathcal{E}}(xy) \le 2$, so that

$$N_{\mathcal{E}}(\bar{\eta}(xy)) = N_{\mathcal{E}}(xy) - N_{\mathcal{E}}(\eta'(xy)) \le 1$$

and $\bar{\eta}(xy) \in \mathcal{E}^+$. We will need this in the following case distinction.

Case 1: If ab is in $\mathcal{E}^+$, then $\varphi(a, b) = (1, ab)$. In this case, $\varphi_1 \varphi_2(a, b, c) = \varphi_1(1, ab, c) = (1, \eta(abc))$ and the application of φ_2 does not change this triple. In particular, the triple $\varphi_2 \varphi_1 \varphi_2(a, b, c)$ contains a 1 on the very left. Furthermore, the triple $\varphi_2 \varphi_1 \varphi_2(a, b, c)$ is totally stable. Since $N_{\mathcal{E}}(abc) \le 2$, we have $\eta(\bar{\eta}(abc)) = (1, \bar{\eta}(abc))$, and this in turn implies

$$\varphi_1 \varphi_2 \varphi_1 \varphi_2(a, b, c) = \varphi_1 \varphi_2(a, b, c) = (1, \eta(abc)) = (\bar{\eta}(\bar{\eta}(abc)), \eta'(\bar{\eta}(abc)), \eta'(abc)).$$

Case 2: Next, assume $ab \notin \mathcal{E}^+$, thus $\eta(ab) = (d, e)$ with $d, e \in \mathcal{E}$. Furthermore, now we assume that $ec \in \mathcal{E}^+$. Then we have

$$\varphi_2 \varphi_1 \varphi_2(a, b, c) = \varphi_2 \varphi_1(d, e, c) = \varphi_2(d, 1, ec) = (1, d, ec).$$

So this triple contains a 1 on the very left. Obviously, such a triple is made totally stable by applying φ_1, and we then obtain

$$\varphi_1 \varphi_2 \varphi_1 \varphi_2(a, b, c) = \varphi_1(1, d, ec) = (1, \eta(dec)) = (1, \eta(abc)).$$

As in the first case, we have here $(1, \eta(abc)) = (\bar{\eta}(\bar{\eta}(abc)), \eta'(\bar{\eta}(abc)), \eta'(abc))$ since $\bar{\eta}(abc)$ has norm at most 1, which proves the last claim for this case.

Case 3: Now we assume $ab \notin \mathcal{E}^+$, $\eta(ab) = (d, e)$ with $d, e \in \mathcal{E}$ and $ec \notin \mathcal{E}^+$. In this case, we have $\eta(ec) = (f, g)$ for some $f, g \in \mathcal{E}$, in particular, $N_{\mathcal{E}}(ec) = 2$. If now df is in $\mathcal{E}^+$, we are done again. So assume that $df \notin \mathcal{E}^+$. Since $d, f \in \mathcal{E}^+$, $N_{\mathcal{E}}(df) \le 2$, and since $df \notin \mathcal{E}^+$, we obtain $N_{\mathcal{E}}(df) = 2$. Thus, the pair $(d, f) = (\overline{ab}, \overline{(ab)'c})$ is geodesic. Furthermore, we have assumed that $((ab)', c) = (e, c)$ is a geodesic pair, so we can apply the condition (F5) of Definition 2.1.2, as reformulated in Lemma 2.1.6. (Recall that factorability of monoids in the sense of Definition 2.1.9 implies, in particular, factorability in the sense of Definition 2.1.2, cf. Theorem 2.1.13.) This implies that $g = ((ab)'c)' = (abc)'$ and $df = \overline{abc}$, so that $\varphi(d, f) = (\overline{\overline{abc}}, (\overline{abc})')$. In particular, the pair (df, g) is η-stable. By the recognition principle 2.1.12 (and Theorem 2.1.13), we know that also $((df)', g)$ has to be η-stable and thus φ-stable. This yields the first claim. Furthermore, we have also shown

$$\varphi_2 \varphi_1 \varphi_2(a, b, c) = \left(\overline{\overline{abc}}, (\overline{abc})', (abc)'\right) = \varphi_1 \varphi_2 \varphi_1 \varphi_2(a, b, c)$$

in this case. This completes the proof.

$\square$

In the next lemma, we are going to generalize the observation of the last lemma, linking η-normal forms to φ-normal forms. It will be important later in order to show that each factorable monoid has a presentation of a special form. Recall from Lemma 2.1.16 and the preliminary remarks to it that we can obtain an η-normal form of an element of M by iterated application of η to the remainders, and this normal form is everywhere η-stable. We will call this normal form the η-normal form, and the normal form given by φ the φ-normal form, in order to distinguish them until we have shown that they coincide.

Lemma 2.3.9. *Let $(M, \mathcal{E}, \eta)$ be a factorable monoid, and set $\varphi(a, b) = \eta(ab)$ for all $a, b \in \mathcal{E}^+$. Then for any tuple $(a_n, \ldots, a_1)$, the φ-normal form of this tuple equals the η-normal form of the product $[a_n \ldots a_1] \in M$.*

Here, the square brackets stress that we are considering the element of the monoid M represented by this word. For one-letter words and sometimes elsewhere, we will omit the brackets if there is no danger of confusion.

Proof. We proceed by induction on n. The claim is clear for $n = 1$ and $n = 2$, and is proved for $n = 3$ in Lemmas 2.3.5 and 2.3.8.

Now assume the claim is already proved for all tuples of length smaller than n. Since applications of φ_i and deleting the 1's do not affect the product of the tuple, we may assume that $(a_n, \ldots, a_1)$ is already in its φ-normal form, and have only to show that this is also the η-normal form of the product. First, observe that by Corollary 2.3.4, the word $(a_n, \ldots, a_2)$ is also in its φ-normal form. By the induction hypothesis, this is also the η-normal form of the product. In particular, we know that $\eta'([a_n \ldots a_2]) = a_2$. Furthermore, we know that

$$\varphi(a_2, a_1) = \eta(a_2 a_1) = (a_2, a_1)$$

since the φ-normal form is everywhere φ-stable. Now this implies that the pair

$$(\eta'([a_n \ldots a_2]), a_1)$$

is η-stable. Since η satisfies the recognition principle by Theorem 2.1.13, it follows that $([a_n \ldots a_2], a_1)$ is η-stable. This implies that the η-normal form of the product $[a_n \ldots a_2 a_1]$ is given by the normal form of $[a_n \ldots a_2]$, followed by a_1. The η-normal form of $[a_n \ldots a_2]$ is by the induction hypothesis given by $(a_n, \ldots, a_2)$, so the η-normal form of $[a_n \ldots a_2 a_1]$ is $(a_n, \ldots, a_2, a_1)$, which completes the proof. $\square$

We are now ready for the proof of Theorem 2.3.2, due to M.Rodenhausen.

Proof. (of Theorem 2.3.2)

Assume first that we are given a monoid M, a generating system $\mathcal{E}$ and a local factorability structure φ on it. Due to Corollary 2.3.7, we know that each element of the

monoid has a well-defined normal form. For $m = 1$, set $\eta(1) = (1,1)$. If $m \neq 1$ and $\mathrm{NF}(m) = (a_n, \ldots, a_2, a_1)$, define

$$\eta(m) = ([a_n \ldots a_2], [a_1]).$$

First, this is well-defined since the normal form is well-defined. It also satisfies

$$\bar\eta(m) \cdot \eta'(m) = m$$

by definition. Next, $\eta'(m)$ is an element of $\mathcal{E}$ unless $m = 1$, where we set $\eta(1) = (1,1)$.

Moreover, observe that the normal form given by a local factorability structure is geodesic. Indeed, since we are allowed to start with any representative of an element $m \in M$ in order to compute the normal form, we can start with one with the minimal number of letters. Since the normal form procedure cannot increase the number of letters, we obtain as a result also a minimal representative for the word. We also know that if $\mathrm{NF}(m) = (a_n, \ldots, a_2, a_1)$, then

$$\mathrm{NF}([a_n \ldots a_2]) = (a_n, \ldots, a_2)$$

by Lemma 2.3.3. In particular, we see that $N_\mathcal{E}(\bar\eta(m)) = n - 1 = N_\mathcal{E}(m) - 1$ for $m \neq 1$. This implies that η is a factorization map.

We are going to use Theorem 2.1.13 and will show that η satisfies (F4) and (F5) of Definition 2.1.2 as well as the recognition principle. So assume $m \in M$ is any element not equal to 1, and let $\mathrm{NF}(m) = (a_n, \ldots, a_1)$ be its normal form. Let furthermore b be some element of $\mathcal{E}$. By Lemma 2.1.6, for (F4) and (F5), we have first to show that, whenever (a_1, b) and $([a_n \ldots a_2], c)$ are geodesic, then so is (m, b), where c is defined by

$$\mathrm{NF}(a_1, b) = \varphi(a_1, b) =: (c, d).$$

So assume (a_1, b) and $([a_n \ldots a_2], c)$ are geodesic. Note that $c \neq 1$ and $d \neq 1$ since $N_\mathcal{E}(a_1 b) = 2$. Now we have already shown that the length n of the φ-normal form of m coincides with the $\mathcal{E}$-word length of m. Recall that computing the normal form of mb, we may start with the word $(a_n, \ldots, a_1, b)$ and observe that the left n letters are in normal form. The normal form procedure tells then to apply $\varphi_n \ldots \varphi_1$ to this tuple. The application of φ_1 turns this tuple into $(a_n, \ldots, a_2, c, d)$, in particular, no 1 appears. But applying $\varphi_n \ldots \varphi_2$ to this tuple corresponds exactly to computing the normal form of $[a_n \ldots a_2] \cdot c$, which does not produce a 1 since the pair $([a_n \ldots a_2], c)$ was assumed to be geodesic. Thus, the application of $\varphi_n \ldots \varphi_1$ to the tuple $(a_n, \ldots, a_1, b)$ does not produce a 1 and so yields the normal form containing $n + 1$ letters. This shows that the norm condition is satisfied.

Furthermore, we stay in the situation where (a_1, b) and $([a_n \ldots a_2], c)$ as defined above are geodesic, and we are going to check that

$$\eta'(\eta'(m)b)\eta'(mb),$$
$$\bar\eta(m) \cdot \bar\eta(\eta'(m)b) = \bar\eta(mb).$$

But we have already seen that the normal form of mb has d as the end of the normal form and $[a_n \ldots a_2] \cdot c$ is going to be the remainder of mb. Altogether, we have:

$$\eta'(\eta'(m)b) = \eta'(a_1 b) = d = \eta'(mb),$$
$$\bar{\eta}(m) \cdot \bar{\eta}(\eta'(m)b) = [a_n \ldots a_2] \cdot c = \bar{\eta}(mb),$$

which shows the conditions of Lemma 2.1.6 are satisfied. Thus, (F4) and (F5) of Definition 2.1.2 hold.

Last, we are going to prove that η satisfies the recognition principle. Let as before m be an element of M with $(a_n, \ldots, a_1)$ as a normal form, and let b be an element of $\mathcal{E}$. Assume first that $\eta(\eta'(m)b) = (\eta'(m), b)$, i.e., (a_1, b) is in normal form and thus stable under φ. This implies in particular that

$$\varphi_1(a_n, \ldots, a_1, b) = (a_n, \ldots, a_1, b).$$

Since $(a_n, \ldots, a_1)$ is a normal form, it also follows by Lemma 2.3.3 that

$$\varphi_n \ldots \varphi_2 \varphi_1(a_n, \ldots, a_1, b) = (a_n, \ldots, a_1, b),$$

so $(a_n, \ldots, a_1, b)$ is the normal form of mb and $\eta(mb) = ([a_n \ldots a_1], b) = (m, b)$, as we wanted to show.

Assume now for the other implication that $\eta(mb) = (m, b)$. Assume $\mathrm{NF}(mb) = (c_k, \ldots, c_1)$. Then by definition, $c_1 = b$, and $[c_k \ldots c_2] = m$, where the word $(c_k, \ldots, c_2)$ is in normal form (using Lemma 2.3.3 again). Since the normal form is unique, we know that the tuples $(c_k, \ldots, c_2)$ and $(a_n, \ldots, a_1)$ have to coincide, so $(a_n, \ldots, a_2, a_1, b)$ is the normal form of mb. It is everywhere φ-stable, in particular,

$$\eta(a_1 b) = \varphi(a_1, b) = (a_1, b).$$

This completes the proof that local factorability induces factorability in the usual sense.

Now we will assume to have a factorable monoid $(M, \mathcal{E}, \eta)$ and we want to construct a local factorability structure for it. As announced before, we define a map

$$\varphi \colon \mathcal{E}^+ \times \mathcal{E}^+ \to \mathcal{E}^+ \times \mathcal{E}^+$$

via $\varphi(a, b) := \eta(a \cdot b)$. This map is idempotent since

$$\bar{\eta}(ab) \cdot \eta'(ab) = ab.$$

Furthermore, it obviously satisfies $\varphi(a, 1) = (1, a)$. The stability for triples (i.e., Property 3 of the Definition 2.3.1) is satisfied due to Lemma 2.3.8.

Now we are going to check the fifth condition of Definition 2.3.1, the normal form condition. Note that by Lemma 2.3.8, we know that the φ-normal form of the triple (a, b, c) is given by $(\bar{\eta}(\bar{\eta}(abc)), \eta'(\bar{\eta}(abc)), \eta'(abc))$, so coincides with the η-normal form of the element of M given by the product abc. Since applying φ_1 to (a, b, c) does not change this product in M, the fifth condition is satisfied.

Now we have to show that the first condition of Definition 2.3.1 is satisfied, i.e., that the monoid M admits a presentation of the form

$$\langle \mathcal{E} | a \cdot b = \overline{ab} \cdot (ab)' \text{ for all } a, b \in \mathcal{E} \rangle.$$

Observe first that M surely admits a surjective map ψ from the monoid

$$M' = \langle \mathcal{E} | a \cdot b = \overline{ab} \cdot (ab)' \text{ for all } a, b \in \mathcal{E} \rangle,$$

given by identity on the generators. Next, observe that we already showed that φ turns the monoid M' into a local factorable monoid. By Lemma 2.3.9, the η-normal form of the image $\psi(x)$ of some element $x \in M'$ coincides with the φ-normal form of this element x, which implies that ψ is also injective. This completes the proof of the theorem.

$\square$

We want to emphasize the following corollary since it is important for us and also a bit surprising when first looking at the definition of factorability.

Corollary 2.3.10. *([53]) Let $(M, \mathcal{E}, \eta)$ be a factorable monoid. Then it admits a presentation of the form*

$$\langle \mathcal{E} | a \cdot b = \overline{ab} \cdot (ab)' \text{ for all } a, b \in \mathcal{E} \rangle.$$

The following observation will be useful while checking the conditions of Definition 2.3.1 to be satisfied.

Lemma 2.3.11. *Let $\mathcal{E}$ be a set. Let*

$$\varphi \colon \mathcal{E}^+ \times \mathcal{E}^+ \to \mathcal{E}^+ \times \mathcal{E}^+$$

be a map with $\varphi^2 = \varphi$ and $\varphi(a, 1) = (1, a)$ and satisfying stability for triples. (In other words, we start with any set $\mathcal{E}$ and a map φ satisfying the second, third and fourth conditions of Definition 2.3.1.) Furthermore, we define the normal form function $\mathrm{NF} \colon \mathcal{E}^* \to \mathcal{E}^*$ as in Definition 2.3.1. Then the fifth condition of Definition 2.3.1 is satisfied for a triple (a, b, c) whenever (a, b) or (b, c) is a stable pair.

Proof. If (b, c) is a stable pair, the statement is clear. If (a, b) is stable, then by Lemma 2.3.5 we have

$$\mathrm{NF}(a, b, c) = \varphi_1 \varphi_2 \varphi_1 \varphi_2(a, b, c) = \varphi_1 \varphi_2 \varphi_1(a, b, c).$$

Furthermore, we know by Lemma 2.3.8 that the resulting triple is everywhere stable, so we obtain

$$\varphi_1 \varphi_2 \varphi_1(a, b, c) = \varphi_1 \varphi_2 \varphi_1 \varphi_2 \varphi_1(a, b, c) = \mathrm{NF}(\varphi_1(a, b, c)).$$

This yields the claim.

$\square$

Last, we want to show a corollary to the Criterion 2.3.2, which is sometimes convenient.

Corollary 2.3.12. *Let M be a monoid, $\mathcal{E}$ a generating system and*

$$\varphi \colon \mathcal{E}^+ \times \mathcal{E}^+ \to \mathcal{E}^+ \times \mathcal{E}^+$$

be a map satisfying:

1. *$M \cong \langle \mathcal{E} | (a, b) = \varphi(a, b) \rangle$.*

2. *Idempotency: $\varphi^2 = \varphi$.*

3. *Value on norm 1 elements: $\varphi(a, 1) = (1, a)$.*

4. *The equality $\varphi_1 \varphi_2 \varphi_1 \varphi_2 = \varphi_2 \varphi_1 \varphi_2 = \varphi_2 \varphi_1 \varphi_2 \varphi_1$ holds evaluated on every triple in $\mathcal{E}$.*

Then φ is a local factorability structure, and the monoid M is factorable with respect to the generating system $\mathcal{E}$.

Proof. First, note that $\varphi_2 \varphi_1 \varphi_2$ makes a totally stable triple out of any triple: It is stable under φ_2 since $\varphi_2^2 = \varphi_2$ and it is stable under φ_1 since $\varphi_1 \varphi_2 \varphi_1 \varphi_2 = \varphi_2 \varphi_1 \varphi_2$. In particular, $\varphi_2 \varphi_1 \varphi_2$ yields in this situation always an extended normal form. Thus, $\varphi_2 \varphi_1 \varphi_2 = \varphi_2 \varphi_1 \varphi_2 \varphi_1$ implies $\mathrm{NF} \circ \varphi_1 = \mathrm{NF}$ when applied to triples. This completes the proof. $\square$

3 Rewriting Systems and Discrete Morse Theory

In this chapter, we collect the basic definitions and facts about rewriting systems and discrete Morse theory. Discrete Morse theory comes in two flavors: an algebraic version for chain complexes and a geometric version for CW complexes. In Section 3.4, we present some tools we will need in order to apply discrete Morse theory to several examples later in this thesis.

3.1 Rewriting System Basics

One of the results of this thesis is joint work with A. Heß and reveals a connection between factorability structures and complete rewriting systems. The basic notions of rewriting systems are the topic of this section. We closely follow the exposition by A. Heß (cf. [42]), which is in turn based on D. Cohen's survey article [22].

The basic idea of a rewriting system is easy: In a monoid presentation with generating set S, we specify a relation not just by a set, but by a pair of two words in the free monoid S^* over S, i.e., an element in $S^* \times S^*$. Thus, we are going to determine a "direction" for each relation, and it must not be applied in the other direction. There are several properties which are desirable when considering such a rewriting system. We will be mostly interested in the notion of a complete rewriting system. In a complete rewriting system, any non-trivial chain of applications of rewriting rules stops after finite time producing a nice normal form. For our purposes, a result of K. Brown ([12]), which relates complete rewriting systems and noetherian matchings, is of particular interest. It will be made more precise in the next section.

Definition 3.1.1. *Let S be a set (sometimes called alphabet) and denote by S^* the free monoid over S. A set of **rewriting rules** $\mathcal{R}$ on S is a set of tuples $(l, r) \in S^* \times S^*$. The string l is called the **left side** and r is called the **right side** of the rewriting rule.*

1. *We introduce a relation on S^* as follows: We say that w **rewrites** to z, denoted by $w \to_{\mathcal{R}} z$, if there exist $u, v \in S^*$ and some rewriting rule $(l, r) \in \mathcal{R}$ such that $w = ulv$ and $z = urv$.*

2. *A word $w \in S^*$ is called **reducible** (with respect to $\mathcal{R}$) if there is some z such that $w \to_{\mathcal{R}} z$. Otherwise, it is called **irreducible** (with respect to $\mathcal{R}$).*

3. *Denote by $\leftrightarrow_{\mathcal{R}}$ the reflexive, symmetric and transitive closure of $\to_{\mathcal{R}}$. Two words w, z over S are called **equivalent** if $w \leftrightarrow_{\mathcal{R}} z$. Set $M = S^* / \leftrightarrow_{\mathcal{R}}$. We then say that $(S, \mathcal{R})$ is a **rewriting system** for the monoid M.*

We now can define complete rewriting systems.

Definition 3.1.2. *Let $(S, \mathcal{R})$ be a rewriting system.*

1. *$(S, \mathcal{R})$ is called **minimal** if the right side r of every rewriting rule $(l, r) \in \mathcal{R}$ is irreducible and if in addition the left side l of every rewriting rule $(l, r) \in \mathcal{R}$ is irreducible with respect to $\mathcal{R} \smallsetminus \{(l, r)\}$.*

2. $(S, \mathcal{R})$ is called **strongly minimal** if it is minimal and if in addition every element $s \in S$ is irreducible.

3. $(S, \mathcal{R})$ is called **noetherian** if there is no infinite sequence

$$w_1 \to_\mathcal{R} w_2 \to_\mathcal{R} w_3 \to_\mathcal{R} \cdots$$

of rewritings. This implies that every sequence of rewritings eventually arrives at an irreducible word.

4. $(S, \mathcal{R})$ is called **convergent** if it is noetherian and if in every equivalence class of $\leftrightarrow_\mathcal{R}$ there is only one irreducible element.

5. A rewriting system is called **complete** if it is strongly minimal and convergent.

As already mentioned, successive rewritings in a complete rewriting system $(S, \mathcal{R})$ induce a normal form, assigning to each $x \in M$ the unique irreducible word in S^* in its equivalence class.

We want to illustrate the notions with some small examples.

Example 3.1.3. *1. The set $S = \{a, b\}$ can be equipped with the rewriting system $aba \to bab$. The word $bab \in S^*$ is then irreducible, the word $abab$ is reducible. This rewriting system is strongly minimal. It is noetherian since each rewriting strictly decreases the number of a's in the word. It is not convergent: We can rewrite $ababa$ to $babba$ and to $abbab$, which are both irreducible. The monoid defined by this rewriting system is isomorphic to the positive braid monoid B_3^+ on 3 strands. (cf. Definition 4.1.7.)*

2. *We can also equip the set $S = \{a, b\}$ with the rewriting system $ab \to ba$. It is not hard to see that this system is complete, and the set of normal forms is given by $b^k a^l$ with $k, l \geq 0$. The monoid given by this rewriting system is just the free abelian monoid on two generators.*

3. *We can equip the set $S = \{a, b, c\}$ with the rewriting system $ab \to bc$, $bc \to ca$. This system is not minimal since there is a rewriting rule ($ab \to bc$), the right-hand side of which is reducible due to $bc \to ca$. The monoid defined by this rewriting system is a special case of Birman-Ko-Lee monoids, described in Definition 4.7.2. These are closely related to the braid groups.*

4. *We consider the set $S = \{a, b, c\}$ with the rewriting system $ab \to bc$, $ab \to ca$. This system is again not minimal, since the left side ab of the rewriting rule $ab \to bc$ is reducible even if we delete this rewriting rule. Note that this rewriting system defines the same monoid as the one in the last point.*

5. *An easy example of a rewriting system which is not noetherian is given by a rewriting system $a \to a^2$ on the single generator a.*

3.2 Discrete Morse Theory Basics

Later on, we will need techniques from discrete Morse theory. We introduce an algebraic version here. Again, we follow the exposition of A. Heß. (see also [42]).

A **based chain complex** is a non-negatively graded chain complex (C_*, ∂), where each C_n is a free $\mathbb{Z}$-module, together with a choice of basis Ω_n for each C_n. In what follows, $(C_*, \Omega_*, \partial)$ will always be a based chain complex.

We equip each C_n with the inner product $\langle \_, \_ \rangle \colon C_n \times C_n \to \mathbb{Z}$ obtained by regarding Ω_n as an orthonormal basis for C_n. For elements $x \in C_n$ and $y \in C_{n-1}$, we say that $\langle \partial x, y \rangle$ is their **incidence number**. If x, y have the "wrong" dimensions, i.e., if $x \in C_n$, but $y \notin C_{n-1}$, then we set their incidence number $\langle \partial x, y \rangle$ to be zero.

Definition 3.2.1. *A $\mathbb{Z}$-compatible **matching** on a based chain complex $(C_*, \Omega_*, \partial)$ is an involution $\mu \colon \Omega_* \to \Omega_*$ satisfying the following property: For every $x \in \Omega_*$ which is not a fixed point of μ, we have $\langle \partial x, \mu(x) \rangle = \pm 1$ or $\langle \partial \mu(x), x \rangle = \pm 1$. (This last condition is going to be called $\mathbb{Z}$-compatibility.)*

*The fixed points of a matching $\mu \colon \Omega_* \to \Omega_*$ are called **essential**. If $x \in \Omega_n$ is not a fixed point, then $\mu(x) \in \Omega_{n-1} \cup \Omega_{n+1}$. We say that x is **collapsible** if $\mu(x) \in \Omega_{n-1}$, and it is called **redundant** if $\mu(x) \in \Omega_{n+1}$.*

Remark 3.2.2. *Note that it is enough to check $\langle \partial \mu(x), x \rangle = \pm 1$ for redundant cells in order to check that an involution $\mu \colon \Omega_* \to \Omega_*$ is $\mathbb{Z}$-compatible if we know that all non-fixed points are either collapsible or redundant. Indeed, let $x \in \Omega_n$ be a non-fixed point of an involution μ as above. We have to show that $\langle \mu(x), \partial x \rangle = \pm 1$ for the case that x is collapsible. In this case, the image $\mu(x) \in \Omega_{n-1}$ is redundant since $\mu(\mu(x)) = x$ is in Ω_n. So we know that for $y = \mu(x)$, we have $\langle \partial \mu(y), y \rangle = \pm 1$. Inserting $y = \mu(x)$, we obtain $\langle \mu(x), \partial x \rangle = \pm 1$.*

Let μ be a matching on $(C_*, \Omega_*, \partial)$. For two redundant basis elements $x, z \in \Omega_*$ set $x \vdash z$ to be the relation "z occurs in the boundary of the collapsible partner of x", i.e. of $\langle \partial \mu(x), z \rangle \neq 0$.

Definition 3.2.3. *A matching on a based chain complex is called **noetherian** if every infinite chain $x_1 \vdash x_2 \vdash x_3 \vdash \ldots$ eventually stabilizes.*

Given a noetherian matching μ on $(C_*, \Omega_*, \partial)$, we define a linear map $\theta^\infty \colon C_* \to C_*$ as follows. Let $x \in \Omega_*$. If x is essential, we set $\theta(x) = x$. If x is collapsible, we set $\theta(x) = 0$, and if x is redundant we set $\theta(x) = x - \varepsilon \cdot \partial \mu(x)$, where $\varepsilon = \langle \partial \mu(x), x \rangle$.

Note that, if x is redundant, then $\langle x, \theta(x) \rangle = 0$. It is now not hard to check that for every $x \in \Omega_*$ the sequence $\theta(x), \theta^2(x), \theta^3(x), \ldots$ stabilizes (cf. also [42], Section 1.1), and we define $\theta^\infty(x) := \theta^N(x)$ for N large enough. We linearly extend this map to obtain $\theta^\infty \colon C_* \to C_*$.

We can now state the main theorem of discrete Morse theory.

Theorem 3.2.4 (Brown, Cohen, Forman)**.** *Let $(C_*, \Omega_*, \partial)$ be a based chain complex and let μ be a noetherian matching. Denote by $C_*^\theta = \operatorname{im}(\theta^\infty \colon C_* \to C_*)$ the θ-invariant*

chains. Then $(C_*^\theta, \theta^\infty \circ \partial|_{\text{im}(\theta^\infty)})$ is a chain complex, and the map

$$\theta^\infty \colon (C_*, \partial) \longrightarrow \left(C_*^\theta, \theta^\infty \circ \partial|_{\text{im}(\theta^\infty)}\right)$$

is a chain homotopy equivalence. A basis of C_*^θ is given by the essential cells.

For a proof see e.g. [36].

As already announced, there is an interesting connection between complete rewriting systems and noetherian matchings. The following theorem is due to K. Brown ([12]):

Theorem 3.2.5 (Brown [12])**.** *Let M be a monoid given by a complete rewriting system $(S, \mathcal{R})$. Then there exists a noetherian matching on the bar complex with essential cells $[x_n|\ldots|x_1]$ given by the following conditions: If $w_i \in S^*$ is the irreducible representative of x_i, then we require*

(a) $w_1 \in S$,

(b) *The word $w_{i+1}w_i$ is reducible for every $1 \le i \le n-1$,*

(c) *For every $1 \le i \le n-1$, any proper (right) prefix of $w_{i+1}w_i$ is irreducible.*

3.3 Discrete Morse Theory for Graded CW-complexes

In this section, we recollect a version of discrete Morse theory due to E. Batzies ([4]). We follow his exposition very closely. This version can be applied to infinite CW-complexes; furthermore, Batzies formulates the theory in the language of acyclic matchings (instead of discrete Morse functions) which seems to be convenient for our approach. This section is quite technical, and the intuition to keep in mind is the same as in the last section. The reader may also want to skip this section in the beginning and return to it whenever discrete Morse theory for CW-complexes is needed. We start with the basic definitions.

Definition 3.3.1. *Let X be a CW-complex, and let $X^{(*)}$ be the set of its open cells. For two cells $\sigma, \sigma' \in X^{(*)}$, we write $\sigma \le \sigma'$ iff the closed cell $\overline{\sigma}$ is a subset of a closed cell $\overline{\sigma'}$, and call σ a **face** of σ'. We say that a cell σ is a **facet** of a cell σ' if $\sigma \ne \sigma'$, $\sigma \le \sigma'$ and for any $\tau \in X^{(*)}$ with $\sigma \le \tau \le \sigma'$ we have either $\sigma = \tau$ or $\sigma' = \tau$.*

*If $(P, \preceq)$ is any poset, a P-**grading** on X is a poset map $f \colon X^{(*)} \to P$. Given a P-grading f and $p \in P$, we write $X_{\preceq p}$ for the sub-CW-complex of X consisting of all cells σ with $f(\sigma) \preceq p$.*

Definition 3.3.2 ([36])**.** *Let X be a CW complex, let σ be an n-dimensional cell of X and let τ be an $(n+1)$-dimensional cell with characteristic map $f_\tau \colon D^{n+1} \to X$. Assume $\sigma \le \tau$. We call σ a **regular face** of τ if f_τ restricted to $f_\tau^{-1}(\sigma)$ is a homeomorphism onto σ and, in addition, $\overline{f_\tau^{-1}(\sigma)}$ is a closed n-ball in D^{n+1}.*

Remark 3.3.3. *Here, for the definition of regular faces, we use the terminology by Forman ([36]). It seems to us that the Definition 3.1.1 of [4] is at this point incomplete; based on the Proposition 3.2.11 of [4], we conjecture that the author uses in fact the same definition as Forman.*

Definition 3.3.4. *Let X be a CW-complex. The **cell graph** G_X of X is a directed graph with $X^{(*)}$ as the set of vertices and edges given by*

$$E_X := \{\sigma \to \tau | \tau \text{ is a facet of } \sigma\}.$$

*A **matching** on X is a subset $A \subset E_X$ such that the following conditions hold:*

(M1) If $(\sigma \to \tau) \in A$, then τ is a regular face of σ.

(M2) Each cell of X occurs in at most one edge of A.

We associate to a matching A a new graph G_X^A with same vertices and with the edge set

$$E_X^A := (E_X \setminus A) \cup \{\sigma \to \tau | (\tau \to \sigma) \in A\},$$

*i.e., we invert all arrows in A and keep all other arrows unchanged. A matching A is called **acyclic** if we in addition have*

(M3) The graph G_X^A contains no cycle.

*A cell of X is called A-**essential** if it does not occur in A. We denote by $X_{ess}^{(*)}$ the set of essential cells of X.*

Such a matching defines now a new poset which we will need:

Definition 3.3.5. *Let X be a CW-complex and A an acyclic matching on it. We set $A^{(*)} = A \cup X_{ess}^{(*)}$. We define a partial order on $A^{(*)}$ as follows: Let $\widetilde{G}_X^A$ be a graph with vertices $X^{(*)}$ and edge set*

$$E_X^A := E_X \cup \{\sigma \to \tau | (\tau \to \sigma) \in A\},$$

i.e., we add to G_X all reversed edges of A. For $a, b \in A^{(*)}$, we set $a \preceq_A b$ if there is a path in $\widetilde{G}_X^A$ from b to a. If b is an element of the form $\sigma \to \tau$, this means that the path may start either from σ or from τ; similarly, if a is of the form $\sigma \to \tau$, the path may end either at σ or at τ.

*This defines a partial order on $A^{(*)}$. We call the poset $(A^{(*)}, \preceq_A)$ the **matching poset** of A.*

The map given by

$$\begin{array}{rcl} X^{(*)} & \to & A^{(*)} \\ \sigma & \mapsto & \begin{cases} \sigma, & \text{if } \sigma \in X_{ess}^{(*)}, \\ (\tau \to \tau'), & \text{if } \sigma \in \{\tau, \tau'\} \text{ and } (\tau \to \tau') \in A \end{cases} \end{array}$$

*can be checked to be order-preserving. We call it the **universal A-grading** on X.*

We will need some finiteness conditions to handle our CW complexes, which are often not finite dimensional.

Definition 3.3.6. *Let $(P, \preceq)$ be a poset and $f \colon X^{(*)} \to P$ a grading on a CW complex X. We call the grading f **compact** if $X_{\preceq p}$ is compact for all $p \in P$.*

Last, we need the definition of the Morse complex of a matching. It is quite technical. We will still cite it here since we will need it quite explicitly.

Definition 3.3.7. *Let X be a CW complex and A an acyclic matching on it such that the universal A-grading is compact. For all $a \in A^{(*)}$, we define first inductively $(X_A)_{\preceq a}$, and also a map $H(A)_{\preceq a} \colon X_{\preceq a} \to (X_A)_{\preceq a}$. These are going to be the pieces of the Morse complex and of the homotopy equivalence from X to the Morse complex.*

First, if $a \in A^{(*)}$ is minimal, we know that $a \in X_{ess}^{(*)}$ and $X_{\preceq a} = a$. We define $(X_A)_{\preceq a}$ to be equal to a and the map $H(A)_{\preceq a}$ to be just the identity.

Now take any $a \in A^{(*)}$ and suppose the associated piece of the Morse complex $(X_A)_{\preceq b}$ and the Morse equivalence $H(A)_{\preceq b}$ are already constructed for all $b \prec a$ in a way such that $b \preceq b' \prec a$ induces an inclusion of the associated pieces of the Morse complex and the restrictions of the future Morse equivalence are compatible with these inclusions. Then define first

$$(X_A)_{\prec a} := \bigcup_{b \prec a} (X_A)_{\preceq b}$$

to be the colimit over the poset $\{b \prec a\}$ of already known pieces and let the map $H(A)_{\prec a}$ from $X_{\prec a}$ be induced by the already known pieces. Now we have to distinguish whether a is an element of A or of $X_{ess}^{(*)}$. If $a = (\tau \to \sigma) \in A$, then we define $(X_A)_{\preceq a} = (X_A)_{\prec a}$ and let the map be defined by

$$H(A)_{\preceq a} = H(A)_{\prec a} \circ \widetilde{h}_{\tau \to \sigma}$$

where the map $\widetilde{h}_{\tau \to \sigma}$ deforms $X_{\preceq a}$ into $X_{\prec a}$ by deforming $\overline{\tau}$ into the union of its faces different from σ. This is possible since σ is a regular face of τ; for more details, we refer again to [4].

Now we consider the other case $a = \sigma \in X_{ess}^{(*)}$, where σ is a cell of dimension i with characteristic map $f_\sigma \colon D^i \to X_{\preceq a}$. We define

$$(X_A)_{\preceq a} = D^i \cup_{H(A)_{\prec a} \circ f_{\partial \sigma}} (X_A)_{\prec a}$$

so we glue a new cell to $(X_A)_{\prec a}$ via $H(A)_{\prec a} \circ f_{\partial \sigma}$. The new piece of map is now induced by the identity on the new cell: Define

$$H(A)_{\preceq a} = \mathrm{id}_{D^i} \cup_{f_{\partial \sigma}} H(A)_{\prec a}.$$

Last, define the **Morse complex** X_A *to be the colimit of all pieces and the **Morse equivalence** $H(A) \colon X \to X_A$ to be the induced map on it.*

We will need the following theorem which is a version of main theorem of discrete Morse theory in Batzies' flavor.

Theorem 3.3.8. *([4]) Let X be a CW complex and A an acyclic matching on it such that the universal A-grading is compact. Then the i-cells of the Morse complex X_A are in one-to-one correspondence with the essential cells of A of dimension i. Furthermore, the Morse equivalence $H(A)\colon X \to X_A$ is a homotopy equivalence.*

Last, we will need a criterion to check whether the universal A-grading is compact. We will use the following lemma.

Lemma 3.3.9. *([4]) Let X be a CW complex and A an acyclic matching on it. Furthermore, let P be a poset and let $f\colon X^{(*)} \to P$ be a compact grading on X such that $f(\tau) = f(\sigma)$ holds for all $(\tau \to \sigma) \in A$. Then the universal A-grading is also compact.*

We derive a corollary of Theorem 3.3.8.

Proposition 3.3.10. *Let X be a CW complex and A an acyclic matching on it such that the universal A-grading is compact. Assume furthermore that the essential cells of A form a subcomplex X_{ess} of X, i.e., that if $\sigma \in X_{ess}^{(*)}$ and $\tau \leq \sigma$, then $\tau \in X_{ess}^{(*)}$. Then the inclusion $i\colon X_{ess} \to X$ is a homotopy equivalence.*

Proof. We will show that the composition $H(A) \circ i\colon X_{ess} \to X_A$ is a homotopy equivalence; this will imply the claim. More precisely, we will first show inductively that $(X_{ess})_{\preceq a} = (X_A)_{\preceq a}$ for all $a \in A^{(*)}$ and the map $H(A) \circ i$ is the identity. For $a \in A^{(*)}$ minimal, the statement is clear. Assume we have proven the statement for all $b \prec a$ and we would like to show it for a. If a is of the form $(\tau \to \sigma)$ in A, then $(X_{ess})_{\preceq a} = (X_{ess})_{\prec a} \subset X_{\prec a}$. Note that $\widetilde{h}_{\tau \to \sigma}$ is identity on $(X_{ess})_{\preceq a}$, so that we are done in this case.

Now assume that $a = \sigma$ is an essential cell of dimension n. Let $f_\sigma\colon D^n \to X_{\preceq a}$ be the characteristic map of this cell. Note that by assumption the attaching map $f_{\partial \sigma}$ has its image in $(X_{ess})_{\prec a}$ and $(X_{ess})_{\preceq a} = (X_{ess})_{\prec a} \cup_{f_{\partial \sigma}} D^n$. It is also $(X_A)_{\preceq a}$ by the induction hypothesis and by the definition of X_A. Moreover, the composition of the inclusion with $H(A)_{\preceq a}$ is again the identity. This completes the induction step. (Observe that the compactness of the grading enables the induction arguments.) Taking the union of all $(X_A)_{\preceq a}$, we see that $X_{ess} = X_A$ and $H(A) \circ i$ is the identity.

Altogether, we have shown that $X_{ess} \to X$ is a homotopy equivalence. $\square$

3.4 Geometric Realization

The aim of this section is to make some observations about properties of geometric realization which will be used later on. We start with the following two well-known properties:

Proposition 3.4.1 ([37], Section 4.3)**.** *1. The geometric realization of a simplicial set X is a quotient space of the subset $\coprod X_n^{\#} \times \Delta^n$ of $\coprod X_n \times \Delta^n$, where $X_n^{\#}$ denotes the set of non-degenerate n-dimensional simplices of X.*

  2. For a simplicial set X, each point of the geometric realization $|X|$ has a unique presentation as a pair $(x, \underline{t})$, where x is non-degenerate and $\underline{t} \in \Delta^{\dim x}$ is an inner point.

We prove the following easy consequence:

Lemma 3.4.2. *Let X be a simplicial set with the following property: All faces of a non-degenerate simplex are again non-degenerate. Then there is a homeomorphism*

$$r(X) := \left(\coprod X_n^\# \times \Delta^n / \sim \right) \to |X|$$

where $\sim$ is generated by $(d_i x, (t_0, \ldots, t_{n-1})) \sim (x, (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}))$. Moreover, the projection

$$\coprod X_n^\# \times \Delta^n \to \coprod X_n^\# \times \Delta^n / \sim$$

defines a CW structure on $r(X)$. Furthermore, each element of $r(X)$ has a unique representative of the form $(x, \underline{t})$, where $\underline{t} \in \Delta^{\dim x}$ is an inner point.

Proof. Since both $r(X)$ and $|X|$ are quotient spaces of $\coprod X_n^\# \times \Delta^n$ (using Proposition 3.4.1), it is enough to construct mutually inverse bijections $r(X) \to |X|$ and $|X| \to r(X)$ which are compatible with the quotient maps. Then, by the definition of the quotient topology, both maps are continuous and thus homeomorphisms.

The map $f \colon r(X) \to |X|$ is given by simply regarding an equivalence class $[x, \underline{t}]$ in $r(X)$ as an equivalence class in $|X|$. This is clearly well-defined and compatible with the quotient maps.

For the other direction, we take any $[y, \underline{s}] \in |X|$ and consider its unique representative $[x, \underline{t}]$ as in Proposition 3.4.1. Since $x \in X_m^\#$ for some m, it also defines a point $g([y, \underline{s}])$ in $r(X)$. This gives us again a well-defined map, which is obviously compatible with the quotient maps.

It is also immediate that $fg = \mathrm{id}$. For the other direction, let $[x, \underline{t}] \in r(X)$ and assume $\underline{t}$ is not an inner point of $\Delta^{\dim x}$. Then there is an inner point $\underline{u} \in \Delta^m$ and a sequence of natural numbers $i_1, \ldots, i_k$ such that $\underline{t} = \delta_{i_1} \ldots \delta_{i_k}(\underline{u})$. Then

$$[x, \underline{t}] = [d_{i_k} \ldots d_{i_1}(x), \underline{u}] \in r(X)$$

where $d_{i_k} \ldots d_{i_1}(x)$ is again a non-degenerate simplex by assumption. This shows that also $gf = \mathrm{id}$. Altogether, this proves the first claim.

The second claim is completely analogous to the statement that $|X|$ is a CW complex. The last claim follows immediately from the second part of the Proposition 3.4.1. $\square$

Remark 3.4.3. 1. *Simplicial sets like in Lemma 3.4.2 are said to have **Property A**, like in [49].*

2. *We will from now on identify $r(X)$ and $|X|$ under the conditions of the last lemma since these spaces are then homeomorphic and have the same CW structure.*

In discrete Morse theory, we have to check whether a smaller cell is a regular face of a larger one. We provide for this purpose a regularity criterion for realizations of simplicial sets.

Lemma 3.4.4. *Let Y be a simplicial set fulfilling Property A and let s be a non-degenerate n-simplex in Y. Consider $t = d_i(s)$ for some $0 \le i \le n$. Let σ and τ be cells of $r(Y)$ (as defined above) corresponding to s and t, respectively. If $d_j(s) \ne t$ for all $0 \le j \ne i \le n$, then τ is a regular face of σ.*

Proof. Choose a fixed homeomorphism $\psi \colon D^n \to \Delta^n$ such that $S_{\ge 0}^{n-1}$ is mapped homeomorphically to the i-th side of Δ^n. The map

$$f_\sigma \colon \{s\} \times D^n \xrightarrow{\psi} \{s\} \times \Delta^n \hookrightarrow \coprod_j Y_j^\# \times \Delta^j \twoheadrightarrow r(Y)$$

is the characteristic map of σ. Any point in the (open) cell τ is of the form

$$x = [t, (t_0, \dots, t_{n-1})]$$

with $t_i > 0$ and $(t_0, \dots, t_{n-1}) \in \Delta^n$. Note that this is also the unique representative with an inner point in the second coordinate, as described in Lemma 3.4.2. This point is by definition identified with the point represented by

$$(s, (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1}))$$

of $\{s\} \times \Delta^n$. Assume there is another point with representative of the form $(s, \underline{u})$ which is identified with x. Then $\underline{u}$ cannot be an inner point by the uniqueness statement of Lemma 3.4.2. So we can write $\underline{u} = \delta_{i_1} \dots \delta_{i_k}(\underline{v})$, where $k \ge 1$ and $\underline{v}$ is an inner point of an appropriate simplex. Thus, x has also a representative of the form $(d_{i_k} \dots d_{i_1}(s), \underline{v})$. Using again the uniqueness, we see that $\underline{v} = (t_0, \dots, t_{n-1})$ and $d_{i_k} \dots d_{i_1}(s) = t$. This implies that $k = 1$. By hypothesis of the lemma, $d_{i_1}(s) = t$ implies $i_1 = i$. This implies that f_σ is injective when restricted to $f_\sigma^{-1}(\tau)$, where the last one is the interior of Δ^{n-1} considered as i-th boundary of Δ^n. Thus, the second condition for regularity is already fulfilled. Furthermore, the map $f_\sigma \colon D^n \to \overline{\sigma}$ is an identification. It is a simple observation that for an identification map $q \colon Z \to Z'$ and $B \subset Z'$ open or closed subset, the restriction $q \colon q^{-1}B \to B$ is an identification again (cf. e.g. the textbook by T. tom Dieck [59]). Thus the restriction of f_σ to $f_\sigma^{-1}(\overline{\tau})$ is an identification since $\overline{\tau} \subset \overline{\sigma}$ is closed. We can now apply same argument again since $\tau \subset \overline{\tau}$ is open in $\overline{\tau}$. This completes the proof that f_σ restricted to $f_\sigma^{-1}(\tau)$ is a homeomorphism. $\square$

We will need the following Lemma later to apply our regularity criterion. For a small category $\mathcal{C}$, we denote by $N\mathcal{C}$ its nerve.

Lemma 3.4.5 ([49], Lemma 11)**.** *Let $\mathcal{C}$ be a small category. Then $N\mathcal{C}$ has Property A if and only if the following holds: Whenever $f \colon A \to B$ and $g \colon B \to A$ are morphisms in $\mathcal{C}$ such that $g \circ f = \mathrm{id}_A$, then we already have $A = B$ and $f = g = \mathrm{id}$.*

Remark 3.4.6. *K. Brown ([12]) uses a variant of discrete Morse theory for simplicial sets similar to the one we will use. We stick to the version by Batzies since there we have a more detailed description of the Morse complex.*

We will later need the following lemma linking both flavors of discrete Morse theory introduced so far.

Lemma 3.4.7. *Let M be a cancellative monoid without non-trivial invertible elements. Let ν be a noetherian, $\mathbb{Z}$-compatible matching on its normalized inhomogeneous bar complex $\overline{B}_*(M)$, which is a based complex with basis Ω_* consisting of all tuples $[x_n|\ldots|x_1]$ with $x_i \in M \setminus \{1\}$ for all i. If $[x_n|\ldots|x_1]$ is a ν-collapsible cell, it was already mentioned in [42], Section 1.2.2, that its ν-redundant partner equals some $d_j([x_n|\ldots|x_1])$. We will assume that for ν, this j does not lie in $\{0, n\}$. Then the matching ν can be extended to an acyclic matching on the CW complex BM.*

Proof. Recall that the set of cells of BM is given exactly by the non-degenerated simplices of the nerve NM. This set coincides with the set Ω_* above. Thus, ν determines a set of edges in the corresponding graph of the poset of cells $BM^{(*)}$. For the first condition of Definition 3.3.4, we need to check that every redundant cell is a regular face of its collapsible partner. First, observe that according to Criterion 3.4.5 and since M has no non-trivial invertible elements, the simplicial set NM satisfies the Property A. Thus we may apply the Regularity Criterion 3.4.4. Let $[x_n|\ldots|x_1]$ be a ν-collapsible cell, and its redundant partner equals some $d_j([x_n|\ldots|x_1])$ with $1 \leq j \leq n-1$.

We want to show that $d_i([x_n|\ldots|x_1])$ is not equal to $d_k([x_n|\ldots|x_1])$ for all $i \neq k$ with $\{i, k\} \neq \{0, n\}$. Assume $d_i([x_n|\ldots|x_1]) = d_k([x_n|\ldots|x_1])$ for some $i \neq k$ with $\{i, k\} \neq \{0, n\}$. We may assume that $i < k$. If $i \neq 0$ and $k \neq n$, then this would imply that the i-th entry of $d_k([x_n|\ldots|x_1])$, which is x_i, and the i-th entry of $d_i([x_n|\ldots|x_1])$, which equals $x_{i+1}x_i$, coincide. By right cancellation we get $x_{i+1} = 1$. This is a contradiction since $x_l \neq 1$ for all l. For $i = 0$, we know that $k \neq n$ and the k-th entry of $d_k([x_n|\ldots|x_1])$, which is $x_{k+1}x_k$, and the k-th entry of $d_i([x_n|\ldots|x_1])$, which is x_{k+1}, coincide. By left cancellation, we conclude $x_k = 1$, which is again a contradiction. Last, consider the case $k = n$ (and so $i \neq 0$): The i-th entry of $d_i([x_n|\ldots|x_1])$ is, on the one hand, $x_{i+1}x_i$, and it would be equal to the i-th entry of $d_n([x_n|\ldots|x_1])$, which is x_i. Right cancellation now implies $x_{i+1} = 1$. All in all, this shows that the first condition (M1) of Definition 3.3.4 is satisfied.

The second condition (M2) is clearly satisfied since ν is an involution. The acyclicity (M3) is a consequence of the noetherianity of ν: Assume we have a cycle

$$a_1, a_2, \ldots, a_m = a_1$$

in the graph associated to the matching ν on the vertex set $BM^{(*)}$ as in Definition 3.3.4. Without loss of generality, we may assume a_1 to be a vertex corresponding to a cell of the smallest dimension among $a_1, \ldots, a_m$. Note that each edge in the graph changes the dimension, moreover, the edges decreasing the dimension by 1 are exactly the ones not in the matching, and the edges increasing the dimension by 1 are exactly the inverted edges from the matching. So we know that the dimension of a_2 has to be $\dim(a_1) + 1$, since it is not smaller than $\dim(a_1)$. Thus, a_1 and a_2 have to be some matched pair, i.e., $\nu(a_1) = a_2$. So the cell corresponding to a_1 is redundant, a_2 is collapsible and so any edge starting in a_2 decreases the dimension. Hence, $\dim(a_3) = \dim(a_1)$ is the

smallest dimension in the cycle, so a_3 is different from a_1 and has to be redundant by the same argument. Therefore, a_3 is a redundant boundary of the collapsible partner of the redundant cell a_1. Inductively, we obtain a chain

$$a_1 \vdash a_3 \vdash a_5 \vdash \ldots \vdash a_{2\lfloor \frac{m}{2} \rfloor - 1} \vdash \ldots$$

where $\vdash$ is defined before Definition 3.2.3. By the definition of noetherianity, this chain has to stabilize, contradicting our assumption of starting with a cycle. This completes the proof. $\qquad \square$

4 Garside theory

In this chapter, we investigate the connection between Garside theory and factorability structures. Garside theory provides families of examples of factorable monoids and groups. In particular, we will exhibit two factorability structures on the braid groups.

4.1 Garside theory: Basics

In this section we give a short overview of Garside theory needed in this thesis. Garside theory deals amongst other things with greedy normal forms, for example on monoids, and with consequences arising from the existence of such normal forms. These normal forms will provide a wide class of examples of factorable monoids and groups.

The notion of a Garside monoid arises from the following observation: The properties of the braid monoids used by F. Garside in [38] to solve the word and the conjugacy problems in the braid monoids and braid groups are also present in a wider class of groups. There are several sets of axioms which reflect the most important of those properties (e.g. [24], [29], [28], [27]). We will stick to the definition below, which seems to be most appropriate in our context. First however, we clarify which sort of properties is needed. These are certain conditions on the divisibility order in the monoid.

Definition 4.1.1. *[28] Let M be a monoid and let x, y be elements in M. We say "x is a **left divisor** of y" or, equivalently, "y is a **right multiple** of x", and write $x \preceq y$ if there is a $z \in M$ such that $y = xz$. We write $x \prec y$ if $x \preceq y$ and $x \neq y$. We call M **left noetherian** if there are no infinite descending sequences of the form $\ldots \prec x_3 \prec x_2 \prec x_1$. Symmetrically, we write $x \succeq y$ for y being a right divisor of x.*

A weak form of Garside structure is described by the notion of a locally left Gaussian monoid.

Definition 4.1.2. *[28] A monoid M is called **locally left Gaussian** if it is right cancellative, left noetherian, and any two elements admitting a left common multiple also admit a left least common multiple. The notion of **locally right Gaussian** monoid is defined symmetrically. A monoid is called **locally Gaussian** if it is both left and right locally Gaussian.*

Remark 4.1.3. *A left least common multiple c of two elements a, b of a monoid M is a left common multiple of this elements with the following property: whenever d is a left common multiple of a and b, we have $d \succeq c$. This should not be confused with the notion of left minimal common multiple of a and b, meaning a left common multiple of a and b which is not right-divisible by any other left common multiple of a and b.*

Remark 4.1.4. *This is one of many variants of the Garside theory. It seems not to be used often in more recent papers, yet it fits exactly in our context. Unfortunately, "left locally Gaussian" was also called "right locally Garside" in [26]. We will nevertheless stick to the name above. This notion is closely related to the notion of a preGarside monoid in a recent paper by E. Godelle and L. Paris ([39]). It can also be put in the*

context of the book project by P. Dehornoy, F. Digne, E. Godelle, D. Krammer and J. Michel ([27]).

Definition 4.1.5. *A monoid M is called **atomic** if for any element $m \in M \setminus \{1\}$, the number*

$$\|a\| := \sup\{n \mid \exists\, a_1, \ldots, a_n \in M \setminus \{1\} \text{ such that } m = a_1 \ldots a_n\}$$

is finite.

The following special cases of locally left Gaussian monoids are of particular interest.

Definition 4.1.6. *[24] A monoid M is called **Garside monoid** (resp. **Gaussian monoid**) if it is atomic, cancellative and the following conditions (resp. first of the following two conditions) hold:*

1. *For any two elements x, y in M, their left and right least common multiples and their left and right greatest common divisor exist. (We denote them by l-lcm(x, y), r-lcm(x, y), l-gcd(x, y), r-gcd(x, y), respectively.)*

2. *There is an element $\Delta \in M$, called a **Garside element**, such that the set of the left divisors of Δ coincides with the set of right divisors of Δ, is finite and generates M.*

The braid groups were the inspiration for the notion of Garside groups, so it is not surprising that these and similar groups provide examples for locally Gaussian and Garside monoids.

Definition 4.1.7. *Recall that an **Artin group** is a group given by a group presentation of the form*

$$G(S) = \langle S \mid (st)^{m_{s,t}} = 1 \text{ for all } s \neq t, s, t \in S \rangle,$$

*where $m_{s,t}$ are natural numbers ≥ 2 or infinity, with $m_{s,t} = m_{t,s}$ for all $s \neq t \in S$. Here, $m_{s,t} = \infty$ means that the pair s, t does not satisfy any relation. We can associate to each Artin group a **Coxeter group** $W(S)$ by adding relations $s^2 = 1$ for all $s \in S$; this corresponds to setting the numbers $m_{s,s}$ to be 1. The matrix $M_S = (m_{s,t})_{s,t \in S}$ will be called the **Coxeter matrix** defining $G(S)$ or $W(S)$, and the pair (S, M_S) will be also called the **Coxeter system**. For each Coxeter system, we can define the corresponding **(positive) Artin monoid** by the monoid presentation*

$$M(S) = \langle S \mid \underbrace{sts \ldots}_{m_{s,t}} = \underbrace{tst \ldots}_{m_{s,t}} \text{ for all } s \neq t \in S \rangle.$$

*For later use, we will denote the alternating word $sts \ldots$ with m factors by $\langle s, t \rangle^m$. Note that there are a monoid homomorphism $\pi \colon M(S) \to W(S)$ and a group homomorphism $\pi \colon G(S) \to W(S)$ mapping each generator s to its image in the quotient group. We will call $M(S)$ as well as $G(S)$ or sometimes, by abuse of notation, even S **of finite type** if the associated Coxeter group $W(S)$ is finite.*

Remark 4.1.8. *Artin monoids of finite type turn out to determine the behavior of the corresponding Artin group quite completely, as first shown by E. Brieskorn and K. Saito ([10]). In the same article, they investigate more generally the structure of all Artin monoids. Amongst other things, they show that any Artin monoid is (left and right) locally Gaussian. Moreover, it is a Garside monoid if and only if it is of finite type. This class of examples is one of the most important in this thesis.*

Remark 4.1.9. *There are further examples of Garside and Gaussian monoids as e.g. torus knot groups; see [27] for a detailed account.*

Now, we are going to describe the normal form mentioned above. This is a greedy normal form: Loosely speaking, one tries to split off a generator which is as large as possible. We are now going to make this more precise. Furthermore, we are going to recall that being normal form can be checked locally. The following relation makes it easier to describe the local behavior of the normal forms.

Definition 4.1.10. *[28] Let M be a monoid and let $\mathcal{E}$ be a subset of M. For $x, y \in M$, we say that $x \lhd_\mathcal{E} y$ if every right divisor of xy lying in $\mathcal{E}$ is a right divisor of y.*

We are ready to formulate the main property of the future normal form.

Definition 4.1.11. *[28] Let M be a monoid and $\mathcal{E}$ a subset of M. We say that a finite sequence $(x_p, \ldots, x_1)$ in $\mathcal{E}^p$ is $\mathcal{E}$-**normal** if for $1 \leq i < p$, we have $x_{i+1} \lhd_\mathcal{E} x_i$.*

The following result is closely related to the existence of the normal form:

Lemma 4.1.12. *([28], Lemma 1.7) Let M be a left locally Gaussian monoid, and $\mathcal{E}$ a generating set closed under left least common multiples. Then every element $x \in M \setminus \{1\}$ admits a unique greatest right divisor lying in $\mathcal{E}$. Here, "greatest" means again: Every other right divisor of x lying in $\mathcal{E}$ is a right divisor of the greatest one.*

The following theorem of P. Dehornoy and Y. Lafont ([28]) ensures the existence of normal forms in left locally Gaussian monoids. Before stating it, we will need one more notion. It is closely connected to the notion of least common multiples: the notion of a left and right complement.

Definition 4.1.13. *Let M be a left-cancellative monoid. For any two $x, y \in M$ whose right least common multiple exists, we denote by $x \backslash y$ the unique element such that $\text{r-lcm}(x, y) = x \cdot (x \backslash y)$, called the **right complement** of y in x.*
*Analogously, we define x / y to be the **left complement** in a right-cancellative monoid, i.e., the unique element with $\text{l-lcm}(x, y) = (x / y) \cdot y$.*

From now on, we assume $1 \neq \mathcal{E}$. By abuse of notation, we say $\mathcal{E}$ is closed under left complements if $\mathcal{E} \cup \{1\}$ is closed under left complements.

Theorem 4.1.14. *([28], Proposition 1.9) Let M be a locally left Gaussian monoid, $\mathcal{E}$ a generating subset of M that is closed under left least common multiples and left complements. Then every nontrivial element x of M admits a unique minimal decomposition $x = x_p \ldots x_1$ such that $(x_p, \ldots, x_1)$ is a $\mathcal{E}$-normal sequence.*

Remark 4.1.15. *The original formulations in [28] use right locally Gaussian monoids and left $\mathcal{E}$-normal forms. We use everywhere the mirrored version: We use $\lhd_\mathcal{E}$ as in Definition 4.1.10 symmetrically to $\rhd_\mathcal{E}$ of [28] and our $\mathcal{E}$ is closed under left least common multiples and left complement instead of right least common multiples and right complements in [28]. In what follows, we use the term "normal form" for the right $\mathcal{E}$-normal form of Theorem 4.1.14.*

For a Garside monoid, it is possible to make some conclusions about its group of fractions. Recall that the group of fractions of a monoid M consists of a group G together with a monoid homomorphism $i\colon M \to G$ (often suppressed in the notation) and it is characterized by the following universal property: Whenever G' is a group and $f\colon M \to G'$ is a monoid homomorphism, f factors uniquely through i, i.e., there is a unique group homomorphism $f'\colon G \to G'$ such that $f'i = f$. As usual, such a group is unique up to canonical isomorphism if it exists. One possible construction of the group of fractions is given by taking any monoid presentation of M and considering it as a group presentation.

Note that in general i needs not to be injective. Obviously, a necessary condition for i to be injective is the cancellativity of M. Yet, it was shown already by Malcev in 1937 ([48]) that this is not sufficient. One simple sufficient condition is provided by the Ore criterion: If in a cancellative monoid any two elements admit a left common multiple, then this monoid embeds into a group, which is equivalent to the injectivity of i. Moreover, if the Ore condition is satisfied, any element of the group of fractions can be written in the form $a^{-1}b$ with some $a, b \in M$ (cf. e.g. [21], Section 1.10). In particular, any Garside monoid satisfies the Ore condition and embeds into its group of fractions. Furthermore, for the example of Artin groups and Artin monoids above, the above description shows that an Artin group is exactly the group of fractions of the corresponding Artin monoid. It is clear from facts collected so far that an Artin monoid of finite type embeds into the corresponding Artin group. In the general case, this question was open for a long time and it was answered affirmatively by L. Paris:

Theorem 4.1.16. *([52]) Every Artin monoid injects into the corresponding Artin group.*

The fact that Garside monoids satisfy the Ore condition allows to extend the normal form of a Garside monoid to its group of fractions. This is our next aim. A group which can be written as a group of fractions of a Garside monoid is called **Garside group**. We introduce some notation in the following remark.

Remark 4.1.17. *([24]) Let M be a Garside monoid with a Garside element Δ. We denote the set of left divisors of Δ except for 1 by $\mathcal{D}$, and set $S = \mathcal{D} \cup \{1\}$. Recall that S also coincides with the set of right divisors of Δ. For any $t \in S$, we define $t^* = t\backslash\Delta$ and $^*t = \Delta/t$. Observe that $\Delta = tt^* = \,^*tt$ and $(^*t)^* = t$. We also denote $^*t$ by $\alpha(t)$.*

Furthermore, we denote by φ the extension as an endomorphism $M \to M$ of the map $t \mapsto t^{**}$ on $\mathcal{D}$. Note that for any $x \in M$ we have $\Delta\varphi(x) = x\Delta$. Denote furthermore $\delta = \varphi^{-1}$. Note that $\delta = \alpha^2$ on $\mathcal{D}$.

Proposition 4.1.18 ([24])**.** *Let M be a Garside monoid with a Garside element Δ and let G be its group of fractions. Then each element of G has a unique decomposition $x_p \ldots x_1 y_1^{-1} \ldots y_q^{-1}$ with $y_i, x_i \in \mathcal{D}$ for all i, r-gcd$(x_1, y_1) = 1$ and*

$$\text{r-gcd}(^*x_k, x_{k+1}) = 1$$
$$\text{r-gcd}(^*y_k, y_{k+1}) = 1$$

for all k.

In particular, if $q = 0$, $x_p \ldots x_1 \in M$ and $x_1 = $ r-gcd(x, Δ). The normal form of the monoid elements coincides with the $\mathcal{D}$-normal form described in Theorem 4.1.14.

The condition in this proposition reformulates the "greediness" of the normal form.

For the rest of this section, we collect several facts of Garside theory needed in later proofs. To investigate the behavior of the normal form with respect to products, we begin with the following lemma.

Lemma 4.1.19 ([24])**.** *Let M be a Garside monoid with a Garside element Δ and let x, y be elements in M. Then we have:*

$$\text{r-gcd}(xy, \Delta) = \text{r-gcd}(\text{r-gcd}(x, \Delta)\, y, \Delta)$$

For our proofs, we will need some rules for the operations of left least common multiple and left complement. They are summed up in the following lemma.

Lemma 4.1.20 ([24])**.** *Let S be again the set of left divisors of the Garside element Δ in the Garside monoid M. Let x, y, z be elements of M and $s, t \in S$. Then we have:*

1. $(xy)\backslash z = y\backslash(x\backslash z)$

2. $z\backslash(xy) = (z\backslash x)((x\backslash z)\backslash y)$

3. r-gcd$(st, \Delta) = (\delta(t)\backslash\alpha(s))^*$

4. $st = (\alpha(s)\backslash\delta(t)) \cdot ((\delta(t)\backslash\alpha(s))^*)$

To provide a factorability structure on Garside groups, we need to investigate the properties of the normal form given by Proposition 4.1.18. First, we need to know how it behaves with respect to the norm. This question is answered by R. Charney and J. Meier ([19]):

Lemma 4.1.21 ([19])**.** *Let M be a Garside monoid and G its group of fractions. The normal form (as in Proposition 4.1.18) for an element $g \in G$ is geodesic, i.e., the word-length norm of g with respect to $\mathcal{E} = \mathcal{D} \cup \mathcal{D}^{-1}$ is exactly the length of the normal form of g.*

We will need some further properties of the word length in a Garside group.

Lemma 4.1.22 ([19])**.** *Let M be a Garside monoid, G its group of fractions. If $a \in M$ has the word length k with respect to $\mathcal{D}$ and n is a positive integer, then $a\Delta^{-n}$ has word length $N_{\mathcal{E}}(a\Delta^{-n}) \leq \max\{k, n\}$ and the equality holds if and only if $\Delta \not\preceq a$.*

The normal forms as in Proposition 4.1.18 were already used to describe the homology of Garside groups by R. Charney, J. Meier and K. Whittlesey in [20], as stated more precisely in the next theorem. We will slightly generalize this result in Section 4.8.

Theorem 4.1.23 ([20])**.** *Let M be a Garside monoid and G its group of fractions. Recall that we denote the set of left divisors of Δ except for 1 by $\mathcal{D}$. Define*

$$\mathcal{D}_n = \{[\mu_1|\ldots|\mu_n] \,|\, \mu_i \in \mathcal{D} \text{ for all } i \text{ and } \mu_1\ldots\mu_n \in \mathcal{D}\}$$

Let d be the maximal number of non-trivial factors Δ can be expanded into. Then there is a free resolution of $\mathbb{Z}$ as a trivial $\mathbb{Z}G$-module of the form

$$0 \longrightarrow \mathbb{Z}G^{|\mathcal{D}_d|} \longrightarrow \cdots \longrightarrow \mathbb{Z}G^{|\mathcal{D}_2|} \longrightarrow \mathbb{Z}G^{|\mathcal{D}|} \longrightarrow \mathbb{Z} \longrightarrow 0$$

and the differentials are as in the bar resolution.

We are going to need the technique of word reversing in our proofs. This technique, developed by P. Dehornoy, allows to read off some properties of the monoid from its presentation if this presentation satisfies certain regularity conditions. In particular, it provides a criterion for a monoid being left locally Gaussian. We will use this criterion later to show in Section 4.10 that Thompson monoids are left locally Gaussian. We recall the right set-up for word reversing.

Definition 4.1.24 ([28])**.** *A monoid presentation (X, R) is called **positive** if all relations in R are of the form $u = v$ with $u, v \in X^*$ non-empty, where X^* denotes the free monoid generated by X. A presentation is called **complemented** if it is positive and, for all $\alpha, \beta \in X$, there is at most one relation of the form $u\alpha = v\beta$ in R and no relation of the form $u\alpha = v\alpha$ for $u \neq v$.*

Definition 4.1.25 ([28], [25])**.** *Let (X, R) be a positive monoid presentation. Let X^{-1} be a copy of X, where we denote by $x^{-1} \in X^{-1}$ the element associated to $x \in X$. For w, z words over $X \cup X^{-1}$, we say that w is R-**reversible to** z if we can transform w into z by iteratively deleting subwords uu^{-1} with $u \in X^*$ and replacing subwords of the form uv^{-1} by $v'^{-1}u'$ where u, v are non-empty words in X^* and $v'u = u'v$ is a relation in R.*

We will use the following suggestive notation for the complements inside the presentation.

Definition 4.1.26 ([28])**.** *Let (X, R) be a complemented presentation and let u, v be words over X. Denote by u/v and v/u the unique words over X such that uv^{-1} is R-reversible to $(v/u)^{-1}(u/v)$ if such words exist.*

Remark 4.1.27 ([28])**.** *Let (X, R) be a complemented presentation. For $\alpha, \beta \in X$, the elements α/β and β/α are the unique words u, v such that $u\beta = v\alpha$ is a relation in R if they exist.*

Now we are in the position to state the criterion mentioned above.

Theorem 4.1.28 ([28], [29])**.** *Let (X, R) be a complemented presentation of a monoid satisfying the following conditions:*

1. *There exists a map ν of X^* to the ordinals, compatible with the congruence $\equiv$ generated by R, and satisfying $\nu(uv) \geq \nu(u) + \nu(v)$ for all words u, v in X^* and $\nu(\alpha) > 0$ for all $\alpha \in X$.*

2. *For all $\alpha, \beta, \gamma \in X$, we have*

$$(\alpha/\beta)/(\gamma/\beta) \equiv (\alpha/\gamma)/(\beta/\gamma),$$

   i.e., either both sides exist and coincide or neither of them exists.

Then the monoid with the presentation (X, R) is locally left Gaussian.

Remark 4.1.29 ([28])**.** *The operation $-/-$ on words represents the left complement operation $-/-$ in a locally left Gaussian monoid, which justifies the use of the same notation.*

Remark 4.1.30. *In locally left Gaussian monoid, there are no non-trivial invertible elements: Assume $ab = 1$. Multiplying with a^n on the left yields $a^{n+1}b = a^n$. Hence, for each n, the element a^{n+1} is a left divisor of a^n. Thus, we obtain a chain of left divisors $\ldots \preceq a^3 \preceq a^2 \preceq a^1 \preceq 1$. Since the monoid is left noetherian, this chain has to stabilize. So for some k, we have $a^{k+1} = a^k$ and this implies by right cancellation $a = 1$. This implies also $b = 1$.*

We will need later the following easy lemma.

Lemma 4.1.31. *Let M be a locally left Gaussian monoid, and let $x, y \in M$ have the left least common multiple $ax = by$. Then for any $z \in M$, xz and yz admit a left common multiple which is equal to axz.*

Proof. Since $axz = byz$ is a left common multiple of xz and yz, the elements xz and yz have to admit a left least common multiple. We write it in the form $cxz = dyz$. By right cancellation, we have $cx = dy$, and since $ax = by$ is the left least common multiple of x and y, we have $cx = uax$ for some $u \in M$. By definition, there is then an element t in M such that $axz = tcxz$ holds. Right cancellation implies $a = tc$ and $c = ua$, so $1 = tu$. Since M is left noetherian, this implies $u = t = 1$, hence axz is the left least common multiple of xz and yz. $\qquad\square$

4.2 Factorability Structure on Left Locally Gaussian Monoids

The aim of this section is to exhibit a factorability structure on locally left Gaussian monoids. It will correspond to the normal form we discussed in Section 4.1. First, we give a reformulation of Theorem 4.1.14. Again, we assume $1 \neq \mathcal{E}$. By abuse of notation, we say $\mathcal{E}$ is closed under left complements if $\mathcal{E} \cup \{1\}$ is closed under left complements.

Corollary 4.2.1. *Let M be a left locally Gaussian monoid, $\mathcal{E}$ a generating subset of M that is closed under left least common multiples and left complement. For any $x \in M \setminus \{1\}$, the beginning x_1 of the (unique) $\mathcal{E}$-normal form of x is the greatest right divisor of x lying in $\mathcal{E}$ (greatest with respect to "being right divisor").*

Proof. This follows immediately from the proof of Theorem 4.1.14 in [28]. $\qquad\square$

Lemma 4.2.2. *Let M be a left locally Gaussian monoid, $\mathcal{E}$ a generating subset of M that is closed under left least common multiples and left complements. Let $x \in M \setminus \{1\}$ be any non-trivial element and let $a \in \mathcal{E}$ be a generator. Let $\mathrm{NF}(x) = x_p \ldots x_1$ be the (right) $\mathcal{E}$-normal form as described in Theorem 4.1.14. Let $\mathrm{NF}(xa) = y_q \ldots y_2 y_1$. Then either $x_1 a = y_1$ or there is a generator $z \in \mathcal{E}$ such that*

$$\mathrm{NF}(x_1 a) = z y_1.$$

Proof. Assume $y_1 \neq x_1 a$. We know by the maximality of y_1 that y_1 is a left multiple of a, so there is a $t \in M$ with $y_1 = ta$. Thus t is the left complement y_1/a, and hence $t \in \mathcal{E}$. Furthermore, we have $xa = y_q \ldots y_2 ta$. This implies by cancellation $x \succeq t$. By maximality of x_1, we have $x_1 \succeq t$ and by assumption, $t \neq x_1$. So there is a $z \in M \setminus \{1\}$ such that $x_1 = zt$. As above, $z \in \mathcal{E}$. We already have $x_1 a = zta = zy_1$ and have to show that y_1 is the greatest right divisor of $x_1 a$ lying in $\mathcal{E}$. Now, for any $u \in \mathcal{E}$ with $x_1 a \succeq u$, we also have $xa = (x_p \ldots x_2)x_1 a \succeq u$ and thus, by definition, $y_1 \succeq u$. This proves the statement. $\qquad\square$

This lemma allows us to compare the lengths of normal forms of x and xa for a generator a.

Corollary 4.2.3. *Let M be a left locally Gaussian monoid, $\mathcal{E}$ a generating subset of M that is closed under left least common multiple and left complement. Let $x \in M \setminus \{1\}$ and $a \in \mathcal{E}$. Let $\mathrm{NF}(x) = x_p \ldots x_1$ be the (right) $\mathcal{E}$-normal form of x. Let $\mathrm{NF}(xa) = y_q \ldots y_2 y_1$. Then either $q = p$ or $q = p + 1$.*

Proof. We proceed by induction on p. For $p = 1$, we have $x = x_1$, and the claim follows directly from Lemma 4.2.2. Now assume we have proved the statement for all lengths of normal forms $\leq p - 1$. Set $v = x_p \ldots x_2$. We know that this is the normal form of v by the definition of $\mathcal{E}$-normal sequences. Again, by Lemma 4.2.2 we can conclude that either $x_1 a = y_1$ or there exists a $z \in \mathcal{E}$ such that $\mathrm{NF}(x_1 a) = z y_1$.

In the first case, $xa = vy_1 = y_q \ldots y_2 y_1$, thus $v = y_q \ldots y_2$ and by the uniqueness of normal forms, $q = p$ follows.

In the second case, $\mathrm{NF}(x_1 a) = z y_1$, so we know by the induction hypothesis that vz has a normal form of length either $p - 1$ or p on the one hand. On the other hand, we have

$$vzy_1 = x_p \ldots x_2 z y_1 = x_p \ldots x_2 x_1 a = xa = y_q \ldots y_2 y_1$$

and $\mathrm{NF}(vz) = y_q \ldots y_2$ by cancellation and by the uniqueness of the normal form. So $q - 1 = p - 1$ or $q - 1 = p$, and the claim follows. $\qquad\square$

Now we can apply the previous corollary to show that the $\mathcal{E}$-normal form is geodesic. The following lemma is an analogue of Lemma 4.1.21 by R. Charney and J. Meier.

Lemma 4.2.4. *Let M be a left locally Gaussian monoid, $\mathcal{E}$ a generating subset of M that is closed under left least common multiples and left complements. Then the length p of the normal form $x = x_p \ldots x_1$ coincides with the word length with respect to $\mathcal{E}$, $N_{\mathcal{E}}(x)$, for any $x \in M \setminus \{1\}$.*

Proof. It is clear that $l(\mathrm{NF}(x)) \geq N_{\mathcal{E}}(x)$. For the other inequality, we proceed by induction on $k = N_{\mathcal{E}}(x)$. For $k = 1$, the claim is clearly true. Assume we have proven the statement for all word lengths $\leq k-1$. Let $a_k \ldots a_1$ be a minimal word in $\mathcal{E}$ representing x. Then $v = a_k \ldots a_2$ is also a minimal word, i.e., word of minimal length, and has thus word length $k-1$, so $\mathrm{NF}(v) = v_{k-1} \ldots v_1$ by the induction hypothesis. By Corollary 4.2.3, we know that the normal form of $x = va_1$ is either of length $k-1$ or of length k, so $\leq k$. Hence we are done. $\qquad\square$

We sum up the results of this section in the following theorem.

Theorem 4.2.5. *Let M be a left locally Gaussian monoid, $\mathcal{E}$ a generating subset of M that is closed under left least common multiple and left complement. Then $(M, N_{\mathcal{E}})$ is factorable with $\eta(x) = (x_p \ldots x_2, x_1)$ with $\mathrm{NF}(x) = x_p \ldots x_2 x_1$ for $x \neq 1$ and $\eta(1) = (1, 1)$.*

Proof. The proof will essentially assemble the results of the last lemmas. First, since left locally Gaussian monoids are right cancellative, we can use the original Definition 2.1.2 due to Corollary 2.1.14. By definition, $\eta'(x)$ is an element of $\mathcal{E}$ for $x \neq 1$, and

$$\overline{\eta}(x) \cdot \eta'(x) = x$$

holds. Since $x_p \ldots x_2$ is the normal form of $\overline{\eta}(x)$ for an x with $\mathrm{NF}(x) = x_p \ldots x_2 x_1$, we conclude by Lemma 4.2.4 that $N_{\mathcal{E}}(x) = p$ and $N_{\mathcal{E}}(\overline{\eta}(x)) = p - 1$, so the norm condition is satisfied.

For the conditions (F4) and (F5), we have to compare $\mathrm{NF}(x) = x_p \ldots x_2 x_1$ with $\mathrm{NF}(xa)$ for some $a \in \mathcal{E}$. Write $\mathrm{NF}(xa) = y_q \ldots y_1$ as before. First, by Lemma 2.1.6, we have to show that $N_{\mathcal{E}}(xa) = p + 1$ if (x_1, a) and $(\overline{x}, \overline{x'a}) = (x_p \ldots x_2, z)$ are assumed to be geodesic, where $z \in \mathcal{E} \setminus \{1\}$ is characterized by $\mathrm{NF}(x_1 a) = z y_1$. The case distinction in the proof of Lemma 4.2.3, combined with Lemma 4.2.4, shows that in this case, the $\mathcal{E}$-norm of xa is $p + 1$. Furthermore, we know that in this case

$$(xa)' = y_1 = (x'a)'$$

holds. This completes the proof. $\qquad\square$

4.3 Factorability Structure on Artin Monoids

We are going to describe the factorability structures on Artin monoids in more detail. In particular, we are going to take a closer look at square-free elements which are an

appropriate generating system for factorability. We will rely on the analysis of divisibility in Artin monoids due to E. Brieskorn and K. Saito ([10]). The following results will be needed.

Proposition 4.3.1 ([10], Prop. 2.3)**.** *All Artin monoids are cancellative.*

The following lemma gives a necessary condition for three generators to have a common multiple. (It turns out also to be sufficient.)

Lemma 4.3.2 ([10], §2)**.** *An element z of an Artin monoid $M(S)$ can only be divisible by three different letters $a, b, c \in S$ if the Coxeter matrix $M_{\{a,b,c\}}$ defines a finite Coxeter group.*

The following basic observation is also important for us.

Lemma 4.3.3 ([10], §3)**.** *For two letters a, b, their least common multiple is given by $\langle a, b \rangle^{m_{a,b}} = aba \ldots$ (a product of $m_{a,b}$ factors).*

Furthermore, we will need some results about the square-free elements. Recall that an element of an Artin monoid $M(S)$ is **square-free** if there is no word representing this element and containing a square of a generator s^2 ($s \in S$). We will denote the set of square-free elements in $M(S)$ by QF(S). A classical theorem often attributed to J. Tits or H. Matsumoto indicates the importance of square-free elements (cf. e.g. [45]).

Theorem 4.3.4 (Tits, Matsumoto)**.** *Any two reduced words representing the same element in a Coxeter group $W(S)$ can be transformed into each other only using braid relations, i.e., without increasing the length of the word in between.*

In particular, one obtains a set-theoretic section $\tau \colon W(S) \to \mathrm{QF}(S) \subset M(S)$ of the projection $\pi \colon M(S) \to W(S)$. One can use this to show that $\pi \colon \mathrm{QF}(S) \to W(S)$ is a bijection. Furthermore, τ is not a monoid homomorphism in general, but we have $\tau(uv) = \tau(u)\tau(v)$ whenever $l(uv) = l(u) + l(v)$ holds for the Coxeter length l on $W(S)$.

The following proposition is proven by F. Dinge and J. Michel ([32]), based on [50].

Proposition 4.3.5. *([32]) Let $M(S)$ be an Artin monoid. Then the set of all square-free elements $\mathrm{QF}(S) \subset M(S)$ is closed under left and right least common multiples and left and right complement. In particular, there are $\mathrm{QF}(S)$-normal forms in $M(S)$ like described in Theorem 4.1.14.*

This implies the following corollary concerning factorability structures.

Corollary 4.3.6. *Let $M(S)$ be an Artin monoid. Then there is a factorability structure on $M(S)$ with respect to the generating system $QF(S)$ of all square-free elements. The prefix $\eta'(x)$ of an element $x \in M(S)$ is given by the maximal square-free right-divisor of x.*

In some cases, it is possible to show that this is the smallest subset of $M(S)$ closed under left least common multiples and left complements and containing S. For example, this is surely true for all Artin groups of finite type. (cf. e.g. Section 6.6 of [47]). Yet, this does not hold in general. A counterexample, where such a subset is again finite, is given in Section 4.4.

4.4 Example: A Non-Spherical Artin Group

Let $M_{4,4}$ (resp. $G_{4,4}$) be the monoid (resp. group) given by the presentation

$$\langle a, b, c \,|\, abab = baba, bcbc = cbcb, ac = ca \rangle.$$

The group $G_{4,4}$ is non-spherical, i.e., it is not of finite type. In particular, $M_{4,4}$ contains infinitely many square-free elements.

We are going to describe the smallest generating set of $M_{4,4}$ closed under left least common multiples and left complements.

Lemma 4.4.1. *The smallest generating set of $M_{4,4}$ closed under left least common multiples and left complements is given by*

$$\mathcal{E}^+ = \{1, a, b, c, ab, ba, bc, cb, ac, aba, bab, bcb, cbc, acb,$$
$$baba, cbcb, bcba, babc, ababc, bcbca, ababcb, cbcbab, bcbcabc, ababcba\}.$$

Proof. We will explain the main steps of the proof. First, we start with $S = \{a, b, c\}$. We add to this set all left least common multiples and left complements for pairs in this subset. According to Lemma 4.3.3, the least common multiples are

$$abab = baba, cbcb = bcbc, ac,$$

so that the new left complements are given by

$$aba, bab, cbc, bcb.$$

Observe that a set in an Artin monoid which is closed under left complement and contains the generating set S is also closed under left divisors. Thus we have to add

$$ab, ba, cb, bc.$$

Now we form all possible left least common multiples for pairs in the new set. We rely hereby in particular on the fact that a, b, c cannot have a common multiple according to Lemma 4.3.2. This is applicable since the Coxeter group associated to $M_{4,4}$ is not finite, cf. e.g. the classification of finite Coxeter groups in [45]. We demonstrate our method in some cases: First, we show that c and ab cannot have a common multiple. For symmetry reasons, this implies that a and cb cannot have a common multiple. So assume $xc = yab$ is a common multiple of both, and assume x to have minimal S-length amongst all possible choices for x. Now it is in particular a common multiple of b and c, so there exists a $z \in M$ such that

$$xc = yab = zcbcb.$$

By cancellation, we obtain $ya = zcbc$, which is a common multiple of a and c. This implies there exists a $w \in M$ such that

$$ya = zcbc = wac.$$

Cancelling again on the right, we obtain $zcb = wa$, which is a common multiple of b and a. So we obtain an element $u \in M$ such that

$$zcb = wa = uabab.$$

So we see that $zc = uaba$ and we deduce there is a $v \in M$ such that

$$zc = uaba = vca.$$

This implies in turn $uab = vc$, which yields another common multiple of ab and c. We compute the S-length of v: Because of $vca = zc$, it is given by

$$N_S(v) = N_S(z) - 1.$$

Since z satisfies the equation $xc = zcbcb$, we have $N_S(z) = N_S(x) - 3$, so $N_S(v) = N_S(x) - 4$, which contradicts the minimality assumption on x. Similarly, one proves that c and aba and, by symmetry, a and cbc, do not admit common multiples.

In contrast, we compute some least common multiples which do exist. Recall that by Lemma 4.1.31, we know that the left least common multiple of ab and cb is given by acb. Furthermore, observe that any common multiple of c and ba is in particular a common multiple of a and c, so also a multiple of ca. Thus, using again Lemma 4.1.31, we conclude that

$$\text{l-lcm}(c, ba) = \text{l-lcm}(ca, ba) = bcbca,$$
$$\text{l-lcm}(a, bc) = \text{l-lcm}(ac, bc) = babac.$$

As new left divisors, we obtain $bcba$ and $babc$. Careful analysis shows that these are all elements produced by pairs of old ones. So we continue allowing now pairs containing elements produced in the last step. Lemma 4.1.31 gives us again

$$\text{l-lcm}(babc, cbc) = bcbcabc,$$
$$\text{l-lcm}(bcba, aba) = babacba,$$
$$\text{l-lcm}(bab, cab) = bcbcab,$$
$$\text{l-lcm}(bcb, acb) = babacb.$$

Now we check that the set obtained this way is indeed closed under left least common multiple and left complement, which completes the proof. $\qquad\square$

The structure of the poset $\mathcal{E}^+$ (with respect to "right divisibility") is shown in the following picture.

bcbcab   ababcb

acb

bab   bcb

ab   cb

abab   bcbc

b

aba   cbc

ababcba   bcbcabc

ba   bc

a   c   babc

bcba

ac

bcbca   babac
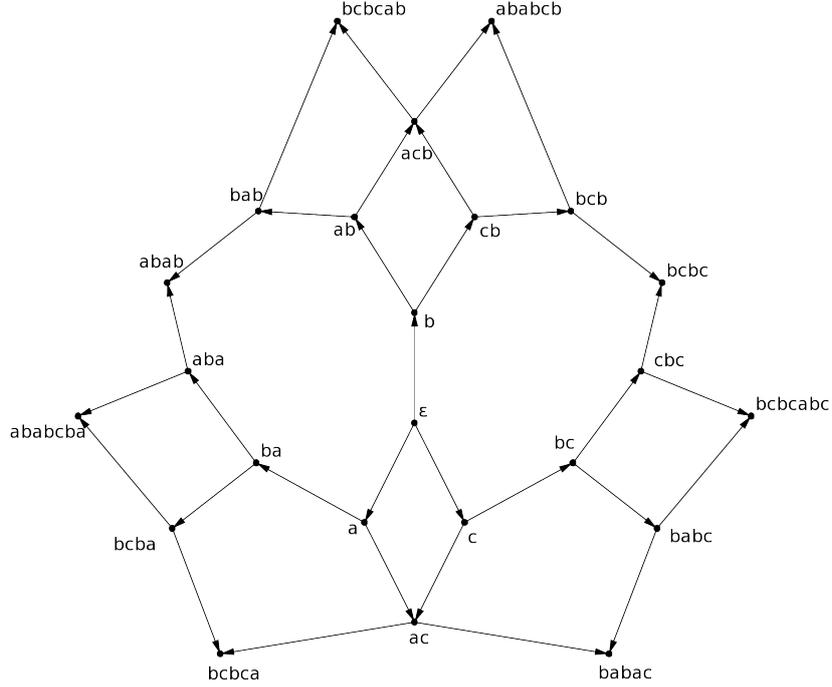
Observe that in this example we have

Proposition 4.4.2. *The inclusion* $(M_{4,4}, \mathcal{E}) \rightarrow (G(M_{4,4}), \mathcal{E} \cup \mathcal{E}^{-1})$ *is* **not** *geodesic; more precisely, there exists an* $x \in M_{4,4}$ *with* $N_{\mathcal{E}}(x) = 3$ *and* $N_{\mathcal{E} \cup \mathcal{E}^{-1}}(x) \leq 2$.

Proof. Consider $x = cbacbc$. We know that the right greedy normal form is a geodesic normal form on $M_{4,4}$ with respect to $\mathcal{E}$, so we can just compute $\eta(x)$ to determine the norm. Observe that cbc is a right divisor of x, so by definition it has to be the right divisor of x'. Note that the only other generators right-divisible by cbc are $bcbc$ and $bcbacbc$. But x cannot be divisible by the last one since the relations of $M_{4,4}$ with respect to $\{a, b, c\}$ are homogeneous and $bcbacbc$ has word length 7 with respect to $\{a, b, c\}$ while $N_{\{a,b,c\}}(x) = 6$.

Assume that there was a $y \in M_{4,4}$ such that $x = y(bcbc)$, then we would have (by cancellation) $cba = yb$, so cba would be a left common multiple of a and b and thus a left multiple of $abab$. Yet, this is impossible (again by considering the $\{a, b, c\}$-word length).

So we have shown $x' = cbc$, $\overline{x} = (cb)a \notin \mathcal{E}$, so $N_{\mathcal{E}}(\overline{x}) = 2$ and the first part of the statement is proven. On the other hand, $x = b^{-1}(bcbacbc)$, so it has at most norm 2 in $G(M_{4,4})$ with respect to $\mathcal{E} \cup \mathcal{E}^{-1}$. This yields the claim. $\square$

The last proposition indicates one of the reasons why it is complicated to get information about the group of fractions of a monoid, even in the relatively simple case of Artin monoids.

4.5 Factorability Structure on Garside Monoids

In a special case of left locally Gaussian monoids, namely in the case of Garside monoids, we are going to show that the factorability structure is particularly nice: Garside monoids are braided factorable. Throughout this section, let M be a Garside monoid and let Δ be a Garside element in M. Furthermore, let $\mathcal{D}$ be the set of its non-trivial left divisors. Here and later in this context, we mean by "non-trivial" all divisors except for 1.

First, we prove the following easy lemma:

Lemma 4.5.1. *Let A be a locally Gaussian monoid, and let $\psi\colon A \to A$ be an automorphism of monoids. Then we have for all $x, y \in A$ such that $\text{l-lcm}(x, y)$ exists:*

$$\psi(\text{l-lcm}(x, y)) = \text{l-lcm}(\psi(x), \psi(y)),$$
$$\psi(x/y) = \psi(x)/\psi(y).$$

Similarly, for all $x, y \in A$ such that $\text{r-lcm}(x, y)$ exists, we obtain

$$\psi(\text{r-lcm}(x, y)) = \text{r-lcm}(\psi(x), \psi(y)),$$
$$\psi(x\backslash y) = \psi(x)\backslash\psi(y).$$

Moreover, for all $x, y \in A$, the identity

$$\text{r-gcd}(\psi(x), \psi(y)) = \psi(\text{r-gcd}(x, y))$$

holds.

Proof. First, we have $\psi(x/y)\psi(y) = \psi(y/x)\psi(x) = \psi(\text{l-lcm}(x, y))$, thus $\psi(\text{l-lcm}(x, y))$ is a left common multiple of $\psi(x)$ and $\psi(y)$. Let now $z \in A$ be any left common multiple of $\psi(x)$ and $\psi(y)$, i.e., $u\psi(x) = v\psi(y) = z$ for some $u, v \in A$. Then we also have

$$\psi^{-1}(u)x = \psi^{-1}(v)y = \psi^{-1}(z).$$

Thus, by definition, $t \cdot \text{l-lcm}(x, y) = \psi^{-1}(z)$ for some $t \in A$ and $\psi(t)\psi(\text{l-lcm}(x, y)) = z$. So any left common multiple of $\psi(x)$ and $\psi(y)$ is right-divisible by $\psi(\text{l-lcm}(x, y))$. This yields the first claim. Since by definition of the left complement we have

$$\text{l-lcm}(\psi(x), \psi(y)) = (\psi(x)/\psi(y))\,\psi(y),$$

and we have just identified the left-hand side with $\psi(\text{l-lcm}(x, y)) = \psi(x/y)\psi(y)$, the second claim follows by right cancellation. The claims about right least common multiples and right complements follow symmetrically.

Last, we want to show that an automorphism ψ preserves right greatest common divisors. By definition, there are some elements $a, b \in A$ so that $x = a \cdot \text{r-gcd}(x, y)$ and $y = b \cdot \text{r-gcd}(x, y)$. So we have $\psi(x) = \psi(a)\psi(\text{r-gcd}(x, y))$, and similarly for y, hence $\psi(\text{r-gcd}(x, y))$ is a right common divisor of $\psi(x)$ and $\psi(y)$. Now let z be any right common divisor of $\psi(x)$ and $\psi(y)$, thus there exist elements $c, d \in A$ with $\psi(x) = cz$ and $\psi(y) = dz$. Applying ψ^{-1}, we obtain $x = \psi^{-1}(c)\psi^{-1}(z)$ and similarly, $\psi^{-1}(z)$ is a

right divisor of y, so it is a right divisor of r-gcd(x, y) since $\psi^{-1}(z)$ is a right common divisor of x and y. This implies the existence of some $t \in A$ with r-gcd$(x, y) = t\psi^{-1}(z)$. Applying ψ again shows that z is a right divisor of $\psi(\text{r-gcd}(x, y))$, so $\psi(\text{r-gcd}(x, y))$ is the right greatest common divisor of $\psi(x)$ and $\psi(y)$. $\qquad\square$

The set $\mathcal{D}$ is closed unter left least common multiples: Since any two elements are right divisors of Δ, so is their left least common multiple. Similarly, $\mathcal{D}$ is closed under left complements. Thus, there is a factorability structure on $(M, \mathcal{D})$ given by Theorem 4.2.5. We will prove now that Garside monoids are even braided factorable (cf. Definition 2.1.17).

Proposition 4.5.2. *The factorability structure from Theorem 4.2.5 on a Garside monoid* (M, Δ) *is braided factorable.*

Proof. The proof works simply by direct, yet tedious computation. Observe that $\delta \colon M \to M$, uniquely described by $\Delta x = \delta(x)\Delta$ for all $x \in M$, is an automorphism of M, thus Lemma 4.5.1 applies. Furthermore, as in Remark 4.1.17, we denote for $s \in \mathcal{D}$ by $\alpha(s)$ the unique element such that $\alpha(s)s = \Delta$. Similarly, s^* is the unique element such that $ss^* = \Delta$. We have $\alpha(s^*) = s$ for all $s \in \mathcal{D} \cup \{1\}$ by definition. We depict the compositions $f_1 f_2 f_1$ and $f_2 f_1 f_2$ schematically below. Let a, b, c be in $\mathcal{D}$.
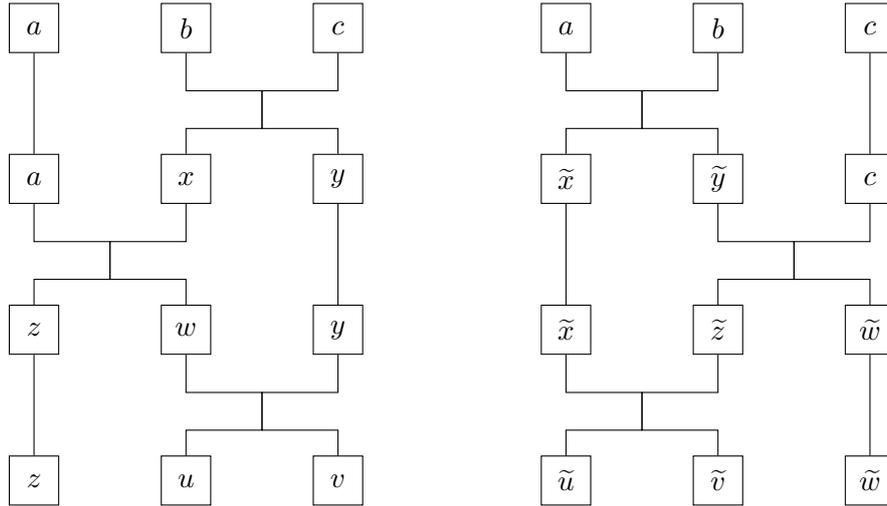
Recall that the factorability structure on Garside monoids is a special case of the factorability structure on left locally Gaussian monoids as described in Theorem 4.2.5. Using Corollary 4.2.1, we know that for any $t \in M \setminus \{1\}$, the prefix $\eta'(t)$ is given by the greatest right divisor of t lying in $\mathcal{D}$. Being the greatest right divisor of t lying in $\mathcal{D}$ can be rephrased as being the greatest right divisor of t which is also a right-divisor of Δ. So we have (cf. also Proposition 4.1.18):

$$\eta'(t) = \text{r-gcd}(t, \Delta).$$

This implies in the picture above: $v = \text{r-gcd}(wy, \Delta)$ and $w = \text{r-gcd}(ax, \Delta)$.

First, we check that $v = \widetilde{w}$. Indeed, we obtain by inserting definitions and using Lemma 4.1.19:

$$v = \text{r-gcd}(\Delta, wy) = \text{r-gcd}(\text{r-gcd}(ax, \Delta)y, \Delta) = \text{r-gcd}(axy, \Delta) = \text{r-gcd}(abc, \Delta)$$

and

$$\widetilde{w} = \text{r-gcd}(\Delta, \widetilde{y}c) = \text{r-gcd}(\text{r-gcd}(ab, \Delta)c, \Delta) = \text{r-gcd}(abc, \Delta).$$

So we are going to verify $u = \widetilde{v}$ and $z = \widetilde{u}$ to establish the statement. Note that by cancellation it is enough to show $z = \widetilde{u}$ since $zuv = abc = \widetilde{u}\widetilde{v}\widetilde{w}$. By Parts 3 and 4 of Lemma 4.1.20, we observe:

$$\begin{aligned}
y &= \text{r-gcd}(bc, \Delta) = (\delta(c)\backslash\alpha(b))^*, \\
x &= \alpha(b)\backslash\delta(c).
\end{aligned}$$

Applying the Lemma 4.1.20 again, we obtain

$$\begin{aligned}
z &= \alpha(a)\backslash\delta(x) \\
&= \alpha(a)\backslash\delta\left(\alpha(b)\backslash\delta(c)\right) \\
&= \alpha(a)\backslash\left(\delta(\alpha(b))\backslash\delta^2(c)\right) \\
&= \left(\delta(\alpha(b)) \cdot \alpha(a)\right)\backslash\delta^2(c),
\end{aligned}$$

where the third equality is an application of Lemma 4.5.1 and the last equality follows from Part 1 of Lemma 4.1.20. Now we are going to compute $\widetilde{u}$ in order to compare it with z. A first application of 4.1.20(4) yields

$$\widetilde{u} = \alpha(\widetilde{x})\backslash\delta(\widetilde{z}).$$

For the same reason, the following identity for $\widetilde{z}$ holds:

$$\widetilde{z} = \alpha(\widetilde{y})\backslash\delta(c).$$

Next, we insert the formula for $\widetilde{z}$ into the one for $\widetilde{u}$ and we get

$$\begin{aligned}
\widetilde{u} &= \alpha(\widetilde{x})\backslash\delta\left(\alpha(\widetilde{y})\backslash\delta(c)\right) \\
&= \alpha(\widetilde{x})\backslash\left(\delta(\alpha(\widetilde{y}))\backslash\delta^2(c)\right) \\
&= \left(\delta(\alpha(\widetilde{y})) \cdot \alpha(\widetilde{x})\right)\backslash\delta^2(c).
\end{aligned}$$

This implies that it is enough to show:

$$\delta(\alpha(\widetilde{y})) \cdot \alpha(\widetilde{x}) = \delta(\alpha(b)) \cdot \alpha(a).$$

By multiplication with $\widetilde{x}$ on the right, we see that this identity is equivalent to

$$\begin{aligned}
\delta(\alpha(b))\alpha(a)\widetilde{x} &= \delta(\alpha(\widetilde{y}))\alpha(\widetilde{x})\widetilde{x} \\
&= \alpha^3(\widetilde{y})\Delta \\
&= \Delta\alpha(\widetilde{y}),
\end{aligned}$$

where we used the definition $\alpha(s)s = \Delta$, as well as $\alpha^2 = \delta$ and $\Delta s = \delta(s)\Delta$ for all $s \in M$. Multiplying both sides by $\widetilde{y}$ on the right, we observe that it is enough to show that

$$\delta(\alpha(b))\alpha(a)\widetilde{x}\widetilde{y} \;=\; \Delta\alpha(\widetilde{y})\widetilde{y} = \Delta^2.$$

Recall that $\widetilde{x}\widetilde{y} = ab$, so we reduced the statement to

$$\Delta^2 = \delta(\alpha(b))\alpha(a)ab.$$

We simplify the right-hand side:

$$\begin{aligned}
\delta(\alpha(b))\alpha(a)ab &= \delta(\alpha(b))\Delta b \\
&= \Delta\alpha(b)b = \Delta^2.
\end{aligned}$$

Thus, the simplified statement is true, and so is the one we started with. This completes the proof.

$\square$

4.6 Factorability Structure on Garside Groups

We have already shown that there is a factorability structure on left locally Gaussian monoids. Yet, it is hard to transfer information about monoids into information about the corresponding groups of fractions in the general case. In this section, we show how this works in the case of Garside monoids. The key ingredient is the normal form described by P. Dehornoy ([24]) as in Proposition 4.1.18. Furthermore, we will use the results by R. Charney and J. Meier concerning the lengths of normal forms.

In this section, let M be a Garside monoid with a Garside element Δ and let $\mathcal{D}$ be the set of its non-trivial left divisors. Let G be the group of fractions of M.

We define now η on G as follows: For any $z \in G$, let $x_p \ldots x_1 y_1^{-1} \ldots y_q^{-1}$ be the normal form of Proposition 4.1.18. We set

$$\begin{aligned}
\eta'(z) &= y_q^{-1} \\
\overline{\eta}(z) &= x_p \ldots x_1 y_1^{-1} \ldots y_{q-1}^{-1}
\end{aligned}$$

in the case that $q \neq 0$ and

$$\begin{aligned}
\eta'(z) &= x_1 \\
\overline{\eta}(z) &= x_p \ldots x_2
\end{aligned}$$

for $q = 0$ (equivalently, for $z \in M$).

Proposition 4.6.1. *The map η turns the normed group $(G, N_{\mathcal{E}})$ into a factorable group. Here, $N_{\mathcal{E}}$ denotes the word-length norm with respect to the generating set $\mathcal{E} = \mathcal{D} \cup \mathcal{D}^{-1}$.*

Before proving the proposition, we need the following lemma varying slightly the normal forms already known in the literature. Since we need to connect these normal forms with the ones in Proposition 4.1.18, we will give a short proof of this lemma.

Lemma 4.6.2. *For any $z \in G$, there is a unique decomposition $z = w_r \ldots w_1 \Delta^{-m}$ with $m \geq 0$ where $w_r \ldots w_1 \in M$ is in its normal form like in Proposition 4.1.18, and $w_r \ldots w_1 \not\succeq \Delta$ if $m > 0$. If the normal form of z described in Proposition 4.1.18 is given by*

$$x_p \ldots x_1 y_1^{-1} \ldots y_q^{-1}$$

and $q > 0$, then $m = q$ and $r \leq p + q$. Furthermore, if in this case z is not a power of Δ, the relation between w_1 and y_q is given by $w_1 = \varphi^{q-1}(y_q^*)$ or, equivalently, $y_q^{-1} = \Delta^{-1} \varphi^{-q}(w_1)$.

Proof. If $q = 0$ (equivalently, $z \in M$), there is nothing to show. So we assume $q > 0$ (equivalently, $z \notin M$).

We start with the normal form of z as in Proposition 4.1.18 and apply the observations of Remark 4.1.17. In particular, we know for each divisor s of Δ that $s^{-1} = s^* \Delta^{-1}$ holds.

$$
\begin{aligned}
z &= x_p \ldots x_1 y_1^{-1} \ldots y_q^{-1} \\
&= x_p \ldots x_1 y_1^* \Delta^{-1} \ldots y_q^* \Delta^{-1} \\
&= x_p \ldots x_1 y_1^* \varphi(y_2^*) \ldots \varphi^{q-1}(y_q^*) \Delta^{-q}.
\end{aligned}
$$

Observe that all $\varphi^{i-1}(y_i^*)$ are divisors of Δ since $\varphi(\Delta) = \Delta$ and y_i^* are divisors of Δ. Yet, it can happen that $\varphi^{i-1}(y_i^*) = 1$. Since φ is a monoid automorphism, this is equivalent to $y_i^* = 1$, and this in turn is equivalent to $y_i = \Delta$. Observe that if $p > 0$, this cannot occur. Indeed, if $y_j = \Delta$, then we have

$$x_p \ldots x_1 y_1^{-1} \ldots y_q^{-1} = x_p \ldots x_2 (x_1 \Delta^{-1}) \varphi^{-1}(y_1^{-1}) \ldots \varphi^{-1}(y_{j-1}^{-1}) y_{j+1}^{-1} \ldots y_q^{-1}$$

which would contradict the fact that this normal form is geodesic (cf. Lemma 4.1.21) since $x_1 \Delta^{-1} \in \mathcal{E} \cup \{1\}$. Now let $p = 0$. Then $z^{-1} = y_q \ldots y_1$ is a right greedy normal form normal form of Proposition 4.1.18. In it, all Δ's are on the very right. Indeed, if an element of M can be written in the form $a \Delta b$, then it is equal to $a \varphi^{-1}(b) \Delta$. Now follows from the definition of greediness that if the normal form of z^{-1} contains a Δ, then $\Delta = \text{r-gcd}(\Delta, z^{-1}) = y_1$. Iterating this argument, we see that all Δ's in the normal form are on the very right.

Denote now by $w_r, \ldots, w_1$ the letters $x_p, \ldots, x_1, y_1^*, \varphi(y_2^*), \ldots, \varphi^{q-1}(y_q^*)$ if $p > 0$ and the letters $\varphi^{j-1}(y_j^*), \ldots, \varphi^{q-1}(y_q^*)$ for $p = 0$ where j is the smallest index such that $y_j \neq \Delta$.

We check that $z \Delta^q$ has $w_r \ldots w_1$ as its normal form. Indeed, the equation

$$\text{r-gcd}(^*x_k, x_{k+1}) = 1$$

holds by definition. Next, we obtain

$$\text{r-gcd}(^*(y_1^*), x_1) = \text{r-gcd}(y_1, x_1) = 1.$$

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At last, we note that Remark 4.1.17 implies $x^* = {}^*\varphi(x)$ and $\varphi(x^*) = \varphi(x)^*$ for all $\Delta \succeq x$. Recall furthermore from Remark 4.1.17 that for all $t \in \mathcal{D}$, the identity $({}^*t)^* = t$ holds. Analogously, also the identity ${}^*(t^*) = t$ holds true. We use this to check

$$\text{r-gcd}\big({}^*\varphi^i(y_{i+1}^*), \varphi^{i-1}(y_i^*)\big) = \text{r-gcd}\big(\varphi^i(y_{i+1}), \varphi^i({}^*y_i)\big) \overset{4.5.1}{=} \varphi^i\big(\text{r-gcd}({}^*y_i, y_{i+1})\big) = 1.$$

So, $x_p \ldots x_1 y_1^* \varphi(y_2^*) \ldots \varphi^{q-1}(y_q^*)$ is a normal form.

Assume now, for contradiction, $x_p \ldots x_1 y_1^* \varphi(y_2^*) \ldots \varphi^{q-1}(y_q^*) \succeq \Delta$ holds. Then we know that $\varphi^{q-1}(y_q^*) = \Delta$ by the second part of Proposition 4.1.18, thus also $y_q^* = \Delta$ and $y_q = 1$ follows. This contradicts the choice of y_i.

The uniqueness of the new normal form now follows from the uniqueness of the normal form in Proposition 4.1.18.

We have defined w_1 to be $\varphi^{q-1}(y_q^*)$. We want to prove the equivalent formulation $y_q^{-1} = \Delta^{-1}\varphi^{-q}(w_1)$. The definition is equivalent to

$$\varphi^{1-q}(w_1) = y_q^*.$$

We can rewrite $y_q y_q^* = \Delta$ to $y_q^* = y_q^{-1}\Delta$. Inserting this into the equation above and multiplying with Δ^{-1}, we obtain

$$y_q^{-1} = \varphi^{1-q}(w_1)\Delta^{-1} = \Delta^{-1}\varphi^{-q}(w_1).$$

This completes the proof. $\qquad\square$

The following easy fact will be used several times in the proof:

Lemma 4.6.3. *Let M be a Garside monoid and let $a, b \in \mathcal{D}$ be two divisors of Δ. Then $a^* \preceq b$ is equivalent to $ab \succeq \Delta$. Indeed, in this case one can even show that there is a $t \in \mathcal{D} \cup \{1\}$ so that $ab = t\Delta$.*

Proof. The statement $a^* \preceq b$ is equivalent to the existence of a $d \in M$ with $a^*d = b$. Due to cancellativity, this is equivalent to the existence of $d \in M$ with $\Delta d = aa^*d = ab$, and this is in turn equivalent to the existence of $d \in M$ with $ab = \Delta d = \varphi^{-1}(d)\Delta$. This proves the first statement.

Now we assume these equivalent conditions hold. Since b is a right-divisor of Δ and d is a right-divisor of b, the element d lies necessarily in $\mathcal{D} \cup \{1\}$. Since φ^{-1} maps divisors of Δ to divisors of Δ, this completes the proof. $\qquad\square$

We are now ready to prove the proposition above.

Proof. (of Proposition 4.6.1) Note that for a non-trivial z with normal form

$$x_p \ldots x_1 y_1^{-1} \ldots y_q^{-1}$$

the element y_q^{-1} is in $\mathcal{D}^{-1}$ and $x_1 \in \mathcal{D}$, so (F3) of Definition 2.1.2 holds. Furthermore, (F1) holds by definition. By Lemma 4.1.21 and since by definition $\overline{\eta}(z)$ has the normal form $x_p \ldots x_1 y_1^{-1} \ldots y_{q-1}^{-1}$ or $x_p \ldots x_2$, we observe that (F2) is fulfilled.

We now proceed to show that (F4) and (F5) are true as well. Note that if α and β from Definition 2.1.2 as described in Remark 2.1.3 coincide, then (F4) and (F5) are satisfied. Indeed, observe that if both compositions always coincide, then the norm condition is automatically fulfilled.

We have to distinguish several cases.

1. First, take $z \in M, s \in \mathcal{D}$. We use Proposition 4.1.18 and Lemma 4.1.19 to see:

$$\eta'(zs) = \text{r-gcd}(zs, \Delta) = \text{r-gcd}(\text{r-gcd}(z, \Delta)s, \Delta) = \eta'(\eta'(z)s).$$

This implies that on such elements, α and β from Definition 2.1.2 as described in Remark 2.1.3 coincide, thus (F4) and (F5) are satisfied.

2. Now assume $z \notin M$, $s \in \mathcal{D}$. Recall that by Lemma 2.1.6, we have to show that if the pairs $(\eta'(z), s)$ and $(\overline{\eta}(z), \overline{\eta}(\eta'(z)s))$ are geodesic pairs, then (z, s) is a geodesic pair and that in this case $\eta'(\eta'(z)s) = \eta'(zs)$ holds. So we assume the pairs $(\eta'(z), s)$ and $(\overline{\eta}(z), \overline{\eta}(\eta'(z)s))$ are geodesic.

   We use the normal form $z = w_r \dots w_1 \Delta^{-q}$, $q > 0$, of Lemma 4.6.2. Observe that we can write

   $$zs = w_r \dots w_1 \Delta^{-q} s = w_r \dots w_1 \varphi^q(s) \Delta^{-q}.$$

   This implies in particular that

   $$N_{\mathcal{E}}(zs) \leq \max\{N_{\mathcal{E}}(w_r \dots w_1 \varphi^q(s)), q\}$$

   by Lemma 4.1.22.

   Now we have to analyze when the pairs $(\eta'(z), s) = (y_q^{-1}, s)$ and $(\overline{\eta}(z), \overline{\eta}(\eta'(z)s))$ are geodesic. We show first that if $r = 0$, the pair $(\eta'(z), s)$ is non-geodesic. If $r = 0$, we have $z = \Delta^{-q}$ and $\eta'(z) = \Delta^{-1}$. This implies

   $$N_{\mathcal{E}}(\eta'(z)s) = N_{\mathcal{E}}(\Delta^{-1}s) = N_{\mathcal{E}}\left((s^*)^{-1}\right) \leq 1.$$

   So we don't need to consider the case $r = 0$.

   Assume from now on $r \geq 1$, i.e., $w_1 \neq 1$. By Lemma 4.6.2, we have

   $$y_q^{-1} = \Delta^{-1}\varphi^{-q}(w_1) = \varphi^{-q+1}(w_1)\Delta^{-1}.$$

   The product $\eta'(z)s$ can thus be written as

   $$\eta'(z)s = \varphi^{-q+1}(w_1)\Delta^{-1}s = \varphi^{-q+1}(w_1\varphi^q(s))\Delta^{-1}.$$

   Since we assumed the pair $(\eta'(z), s)$ to be geodesic, the element $\eta'(z)s$ has length 2. So we use Lemma 4.1.22 to see

   $$2 = N_{\mathcal{E}}\left(\varphi^{-q+1}(w_1\varphi^q(s))\Delta^{-1}\right) \leq \max\{N_{\mathcal{D}}(\varphi^{-q+1}(w_1\varphi^q(s))), 1\}.$$

Thus, we know that the $\mathcal{D}$-length of $\varphi^{-q+1}(w_1\varphi^q(s))$ is at least 2. Since φ is an automorphism of M mapping $\mathcal{D}$ to itself, we conclude that $w_1\varphi^q(s)$ has at least $\mathcal{D}$-length 2. But as a product of two elements of $\mathcal{D}$, it has $\mathcal{D}$-norm exactly 2. In particular, we know that the above inequality is in this case an equality, so by Lemma 4.1.22, the element $\varphi^{-q+1}(w_1\varphi^q(s))$ and thus also $w_1\varphi^q(s)$ is not divisible by Δ.

Denote $\eta(w_1\varphi^q(s))$ by (a,b). Since $w_1\varphi^q(s)$ has norm 2, the pair (a,b) is already the normal form of $w_1\varphi^q(s)$. In particular, a and b are in $\mathcal{D}$, and by the definition of the normal form in Proposition 4.1.18, we have r-gcd$(^*b,a) = 1$. Moreover, since $w_1\varphi^q(s)$ is not divisible by Δ, the prefix b is not equal to Δ. We would like to express $\eta'(\eta'(z)s)$ in a and b using Lemma 4.6.2. Since $\varphi^{-q+1}(w_1\varphi^q(s))$ is not divisible by Δ, the normal form of $\eta'(z)s$ in the sense of Lemma 4.6.2 is given by NF$(\varphi^{-q+1}(w_1\varphi^q(s)))\Delta^{-1}$. Observe that

$$\eta'(z)s = \varphi^{-q+1}(w_1\varphi^q(s))\Delta^{-1} = \varphi^{-q+1}(ab)\Delta^{-1} = \varphi^{-q+1}(a)\varphi^{-q+1}(b)\Delta^{-1}.$$

We check that $\varphi^{-q+1}(a)\varphi^{-q+1}(b)$ is the normal form of $\varphi^{-q+1}(w_1\varphi^q(s))$. We only have to show that $\varphi^{-q+1}(b)$ is the right-most letter of the normal form since the element $\varphi^{-q+1}(w_1\varphi^q(s))$ has norm 2. For this, we use Lemma 4.5.1:

$$\text{r-gcd}(\varphi^{-q+1}(a)\varphi^{-q+1}(b), \Delta) = \varphi^{-q+1}(\text{r-gcd}(ab, \Delta)) = \varphi^{-q+1}(b).$$

So we know by Lemma 4.6.2 that

$$\eta'(\eta'(z)s) = \Delta^{-1}\varphi^{-1}(\varphi^{-q+1}(b)) = \Delta^{-1}\varphi^{-q}(b) = \varphi^{-q+1}(b)\Delta^{-1}$$

and thus $\overline{\eta}(\eta'(z)s) = \varphi^{-q+1}(a)$.

We now look at the second pair. By definition, $\overline{\eta}(z)$ is given by zy_q. We want to compute its normal form as given in Lemma 4.6.2. We have

$$zy_q = w_r\ldots w_1\Delta^{-q}y_q = w_r\ldots w_1\varphi^q(y_q)\Delta^{-q}.$$

We look closer at the product $w_1\varphi(y_q)$ using Lemma 4.6.2 once again:

$$w_1\varphi^q(y_q) = \varphi^{q-1}(y_q^*)\varphi^q(y_q) = \varphi^{q-1}(y_q^*\varphi(y_q)).$$

We can now simplify $y_q^*\varphi(y_q)$ by observing that

$$y_qy_q^*\varphi(y_q) = \Delta\varphi(y_q) = y_q\Delta,$$

which immediately implies $y_q^*\varphi(y_q) = \Delta$. So we obtain in total

$$zy_q = w_r\ldots w_1\varphi^q(y_q)\Delta^{-q} = w_r\ldots w_2\Delta^{-q+1},$$

which is now again a normal form in the sense of Lemma 4.6.2. Here, we use that $w_r\ldots w_2$ is not divisible by Δ since $w_r\ldots w_1$ is not divisible by Δ. Hence, the $\mathcal{E}$-length of $w_r\ldots w_2\Delta^{-q+1}$ is by Lemma 4.1.22 given by $\max\{r-1, q-1\}$.

Since the pair

$$(\bar\eta(z), \bar\eta(\eta'(z)s)) = (w_r \ldots w_2 \Delta^{-q+1}, \varphi^{-q+1}(a))$$

is geodesic, we know that the norm of

$$w_r \ldots w_2 \Delta^{-q+1} \cdot \varphi^{-q+1}(a) = w_r \ldots w_2 a \Delta^{-q+1}$$

must be equal to $\max\{r-1, q-1\} + 1 = \max\{r, q\}$. Combining this with Lemma 4.1.22 applied to $w_r \ldots w_2 a \Delta^{-q+1}$, we see that

$$\max\{r, q\} \le \max\{N_{\mathcal{D}}(w_r \ldots w_2 a), q-1\} \le \max\{r, q-1\}.$$

This, in turn, implies $r = \max\{r, q\} = \max\{r, q-1\}$. So we conclude that the norm of $w_r \ldots w_2 a$ is r and by the second part of Lemma 4.1.22, Δ does not right-divide $w_r \ldots w_2 a$.

Using Lemma 4.1.19 and Proposition 4.1.18 once again, we see that the normal form of $w_r \ldots w_1 \varphi^q(s)$ ends with

$$
\begin{aligned}
\text{r-gcd}(w_r \ldots w_1 \varphi^q(s), \Delta) &= \text{r-gcd}(\text{r-gcd}(w_r \ldots w_1, \Delta)\varphi^q(s), \Delta) \\
&= \text{r-gcd}(w_1 \varphi^q(s), \Delta) = b.
\end{aligned}
$$

In particular, since we have seen $b \ne \Delta$, it follows that Δ does not right-divide $w_r \ldots w_1 \varphi^q(s)$. So the normal form of $w_r \ldots w_1 \varphi^q(s)$ is given by

$$\text{NF}(w_r \ldots w_1 \varphi^q(s)) = \text{NF}(w_r \ldots w_2 a)b.$$

Since $\text{NF}(w_r \ldots w_2 a)$ has length r (e.g. by Lemma 4.2.4), the normal form of $w_r \ldots w_1 \varphi^q(s)$ is of length $r+1$ and the norm of

$$zs = w_r \ldots w_1 \varphi^q(s) \Delta^{-q}$$

is $\max\{q, r+1\} = r+1$. So we have shown that (z, s) is a geodesic pair. Using Lemma 4.6.2, we can compute $\eta'(zs)$: It is given by

$$\Delta^{-1} \varphi^{-q}(b) = \varphi^{-q+1}(b)\Delta^{-1}.$$

This coincides with $\eta'(\eta'(z)s)$. So we have proven the statement for this case.

3. Consider now the case $z \in M$, $s \in \mathcal{D}^{-1}$. There is a $t \in \mathcal{D} \cup \{1\} \setminus \{\Delta\}$ with $s = t\Delta^{-1}$, namely, if $s = u^{-1}$ with $u \in \mathcal{D}$, we have $t = u^*$. Let $z = x_p \ldots x_1$ be the normal form of z. Then $zs = x_p \ldots x_1 t\Delta^{-1}$.

If $x_1^* \preceq t$, then there is by Lemma 4.6.3 an element $c \in \mathcal{D} \cup \{1\}$ such that $x_1 t = c\Delta$. Therefore,

$$\eta'(z) \cdot s = x_1 t\Delta^{-1} = c$$

so the pair $(\eta'(z), s)$ is not geodesic. Thus, (F4) and (F5) of Definition 2.1.2 are satisfied in this case by Lemma 2.1.6.

If $x_1^* \not\preceq t$, then Lemma 4.6.3 implies that $x_1 t \not\succeq \Delta$. We show first that this implies $x_p \dots x_1 t$ is not right-divisible by Δ. Suppose it were, then there is a $u \in M$ so that

$$x_p \dots x_1 t = u\Delta = u \cdot {}^*t t,$$

and by cancellation, ${}^*t \in \mathcal{D}$ is a right-divisor of $x_p \dots x_1$. By definition, x_1 is the greatest divisor of $x_p \dots x_1$ lying in $\mathcal{D}$, so this would imply the existence of some $v \in M$ with $x_1 = v \cdot {}^*t$, and $x_1 t = v \cdot {}^*t t = v\Delta$, contradicting our assumptions.

Hence, $\mathrm{NF}(x_p \dots x_1 t)\Delta^{-1}$ is a normal form in the sense of Lemma 4.6.2. The normal form $\mathrm{NF}(x_p \dots x_1 t)$ ends with

$$\text{r-gcd}(x_p \dots x_1 t, \Delta) = \text{r-gcd}(\text{r-gcd}(x_p \dots x_1, \Delta)t, \Delta) = \text{r-gcd}(x_1 t, \Delta),$$

which we will denote by a. Here, we use Lemma 4.1.19 for the first equality. Thus by Lemma 4.6.2 and by definition of η, we have $\eta'(zs) = \Delta^{-1}\varphi^{-1}(a)$. On the other hand,

$$\eta'(\eta'(z) \cdot s) = \eta'(x_1 t\Delta^{-1}) = \Delta^{-1}\varphi^{-1}(\text{r-gcd}(x_1 t, \Delta)) = \Delta^{-1}\varphi^{-1}(a).$$

This implies that on such elements z and s, the maps α and β from Definition 2.1.2 as described in Remark 2.1.3 coincide, thus (F4) and (F5) are satisfied.

4. Last, we come to the case $z \notin M$, $s \in \mathcal{D}^{-1}$. There is again a $t \in \mathcal{D} \cup \{1\} \setminus \{\Delta\}$ with $s = t\Delta^{-1}$. Let as before $z = w_r \dots w_1 \Delta^{-q}$, $q > 0$, be the normal form of Lemma 4.6.2. Then $zs = w_r \dots w_1 \varphi^q(t)\Delta^{-q-1}$. Again, two cases are possible.

If $w_1^* \preceq \varphi^q(t)$, we have $w_1\varphi^q(t) = d\Delta$ for some $d \in \mathcal{D} \cup \{1\}$ by Lemma 4.6.3. We obtain using Lemma 4.6.2:

$$\eta'(z)s = \Delta^{-1}\varphi^{-q}(w_1)t\Delta^{-1} = \Delta^{-1}\varphi^{-q}(w_1\varphi^q(t))\Delta^{-1} = \Delta^{-1}\varphi^{-q}(d) \in \mathcal{D}^{-1} \cup \{1\}.$$

So $(\eta'(z), s)$ is not geodesic. Thus, (F4) and (F5) of Definition 2.1.2 are satisfied in this case by Lemma 2.1.6.

If $w_1^* \not\preceq \varphi^q(t)$, we conclude again that Δ is not a divisor of $w_r \dots w_1\varphi^q(t)$. Therefore, $\mathrm{NF}(w_r \dots w_1\varphi^q(t))\Delta^{-q-1}$ is the normal form of zs in the sense of Lemma 4.6.2. We compute again as before:

$$\text{r-gcd}(w_r \dots w_1\varphi^q(t), \Delta) = \text{r-gcd}(w_1\varphi^q(t), \Delta) =: a.$$

Thus by Lemma 4.6.2, we have $\eta'(zs) = \Delta^{-1}\varphi^{-q-1}(a) = \varphi^{-q}(a)\Delta^{-1}$. On the other hand, we have to determine $\eta'(\eta'(z) \cdot s)$. First, we can write $\eta'(z)s$ as follows using Lemma 4.6.2:

$$\eta'(z) \cdot s = \varphi^{-q+1}(w_1)\Delta^{-1}s = \varphi^{-q+1}(w_1)\Delta^{-1}t\Delta^{-1} = \varphi^{-q+1}(w_1)\varphi(t)\Delta^{-2}.$$

We now want to compute its prefix. By Lemma 4.6.3, our assumption $w_1^* \not\preceq \varphi^q(t)$ implies $w_1 \varphi^q(t) \not\succeq \Delta$, and thus also

$$\varphi^{-q+1}(w_1)\varphi(t) = \varphi^{-q+1}(w_1 \varphi^q(t))$$

is not divisible by Δ. So $\mathrm{NF}(\varphi^{-q+1}(w_1)\varphi(t))\Delta^{-2}$ is the normal form of $\eta'(z)s$ in sense of Lemma 4.6.2. We use this lemma as well as the Proposition 4.1.18 and Lemma 4.5.1 once again to conclude:

$$\eta'(\eta'(z)s) = \varphi^{-1}\left(\text{r-gcd}(\varphi^{-q+1}(w_1)\varphi(t), \Delta)\right)\Delta^{-1}$$
$$= \varphi^{-q}\left(\text{r-gcd}(w_1 \varphi^q(t), \Delta)\right)\Delta^{-1} = \varphi^{-q}(a)\Delta^{-1}$$

This implies that on such elements z and s, the maps α and β from Definition 2.1.2 as described in Remark 2.1.3 coincide, thus (F4) and (F5) are satisfied. This completes the case distinction.

$\square$

4.7 Braid Groups and Monoids

In this section, we want to use a special case of Proposition 4.3.5 to give a more explicit example of the factorability structures described in the last sections.

Consider the braid monoid B_n^+, $n \geq 1$, i.e., the monoid generated by $n-1$ generators $\sigma_1, \ldots, \sigma_{n-1}$, subject to relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| \geq 2,$$
$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \text{ if } |i-j| = 1.$$

This is a special case of an Artin monoid (cf. Definition 4.1.7), which historically was the prototypical example generalized then to Artin groups and monoids.

On this monoid, we have shown the existence of a factorability structure with respect to the generating system $\mathcal{D}$ of all non-trivial divisors of the Garside element. The Garside element of this monoid can be chosen to be a half-twist, given by the formula

$$\Delta_n = (\sigma_1 \sigma_2 \ldots \sigma_{n-1})(\sigma_1 \sigma_2 \ldots \sigma_{n-2}) \ldots (\sigma_1 \sigma_2)\sigma_1 \in B_n^+.$$

This braid can be drawn as follows:
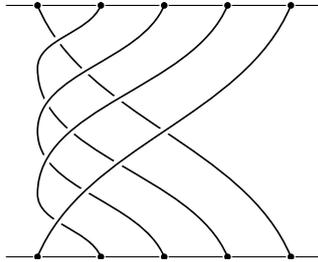
It is shown by E. Brieskorn and K. Saito ([10]) that for Artin monoids of finite type, the square-free elements are exactly the divisors of the standard Garside element Δ. As described in Section 4.3, the square-free elements are in bijection to the corresponding Coxeter group, which in the braid group case is the symmetric group. In particular, we know that $\mathcal{D}$ has $n! - 1$ elements.

The factorability structure on B_n^+ yields by the last section a factorability structure on the braid groups with respect to the generating system $\mathcal{D} \cup \mathcal{D}^{-1}$.

Example 4.7.1. *We want to illustrate the factorability structure by a calculation example. We consider the braid $\sigma_3\sigma_2\sigma_1^{-3}\sigma_2 \in B_4$. We would like to compute the value of η on it. First, we write it in the form xy^{-1} with $x, y \in B_4^+$. Since in B_3^+ we have*

$$\sigma_1\Delta_3 = \sigma_1(\sigma_2\sigma_1\sigma_2) = (\sigma_1\sigma_2\sigma_1)\sigma_2 = \Delta_3\sigma_2,$$

it follows that $\varphi(\sigma_1) = \sigma_2$ holds in B_3^+ and one easily concludes

$$\sigma_1^{-3}(\sigma_2\sigma_1\sigma_2) = (\sigma_2\sigma_1\sigma_2)\sigma_2^{-3}.$$

Using this, we obtain

$$\begin{aligned}
\sigma_3\sigma_2\sigma_1^{-3}\sigma_2 &= \sigma_3\sigma_2\sigma_1^{-3}\sigma_2\sigma_1\sigma_2\sigma_2^{-1}\sigma_1^{-1} \\
&= \sigma_3\sigma_2(\sigma_2\sigma_1\sigma_2)\sigma_2^{-3}\sigma_2^{-1}\sigma_1^{-1} \\
&= \sigma_3\sigma_2^2\sigma_1\sigma_2^{-3}\sigma_1^{-1}.
\end{aligned}$$

Now we compute the normal forms in B_4^+ for $x = \sigma_3\sigma_2^2\sigma_1$ and $y = \sigma_1\sigma_2^3$ (cf. Proposition 4.1.18). One can check that

$$\begin{aligned}
x &= (\sigma_3\sigma_2) \cdot (\sigma_2\sigma_1) \\
y &= (\sigma_1\sigma_2) \cdot \sigma_2 \cdot \sigma_2
\end{aligned}$$

are the normal forms and that $\text{r-gcd}(\sigma_2\sigma_1, \sigma_2) = 1$. Thus,

$$\eta(\sigma_3\sigma_2\sigma_1^{-3}\sigma_2) = (\sigma_3\sigma_2^2\sigma_1\sigma_2^{-2}, \sigma_2^{-1}\sigma_1^{-1}).$$

Instead of the positive braid monoid we may also consider the Birman-Ko-Lee monoids. These are monoids which also have the braid groups as their groups of fractions. They were first defined in [7]. These monoids were shown to be Garside by D. Bessis, F. Dinge and J. Michel ([6]).

Definition 4.7.2. *([7]) The n-th **Birman-Ko-Lee monoid** BKL_n can be defined in terms of generators and relations as follows:*

$$\begin{aligned}
\langle\, a_{ts}, 1 \le s < t \le n | a_{ts}a_{rq} &= a_{rq}a_{ts} \text{ if the intervals } [s,t] \text{ and } [q,r] \text{ are disjoint or nested} \\
a_{ts}a_{sr} &= a_{tr}a_{ts} = a_{sr}a_{tr} \text{ for } 1 \le r < s < t \le n \,\rangle
\end{aligned}$$

The element a_{ts} corresponds to the braid interchanging the t-th and the s-th strand in front of all other strands (cf. the picture of a_{52} with $n = 7$ below).
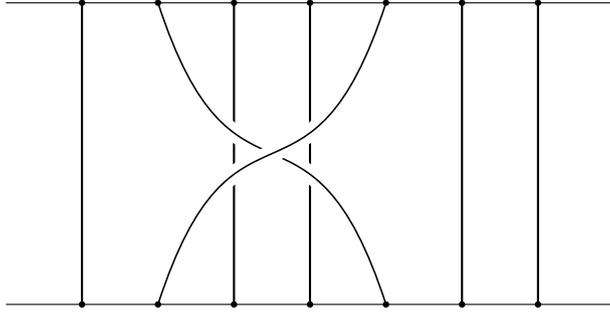
This monoid is Garside with a Garside element (as shown in [6]) given by

$$\delta = a_{n(n-1)}a_{(n-1)(n-2)} \ldots a_{21} = \sigma_{n-1}\sigma_{n-2} \ldots \sigma_1.$$

In particular, the Birman-Ko-Lee monoid BKL_n embeds into the braid group B_n and satisfies the Ore condition. Moreover, it follows from Theorem 4.2.5 that there is a factorability structure on BKL_n with respect to the generating system of all divisors of δ. By the last section, it can be used to define a second factorability structure on the braid groups.

Remark 4.7.3. *It is shown by D. Bessis ([5], Proposition 5.2.1) that the set of right-divisors of δ in BKL_n contains exactly $\frac{1}{n+2}\binom{2n+2}{n+1}$ elements (which is the $(n+1)$-st Catalan number) and is in general smaller than the set of right-divisors of Δ in B_n^+, which has $n!$ elements. So the factorability structure given by the Birman-Ko-Lee monoid has a smaller generating system than the one associated to the positive braid monoid. Note yet that the positive braid monoid seems to be the more "natural" one and was already considered much earlier.*

4.8 Homology of Locally Left Gaussian Monoids

The results of this section were found partially joint with A. Heß. The aim of this section is to generalize Theorem 4.1.23 by R. Charney, J. Meier and K. Whittlesey ([20]) for Garside monoids to left cancellative, locally left Gaussian monoids which are atomic. For this, we give a proof using discrete Morse theory, thus also reproving the original result as already written down in [42].

The proof of the following statement follows closely the proof of Lemma 1.7 of [28].

Lemma 4.8.1. *Let M be a locally left Gaussian monoid, $\mathcal{S}$ a generating set closed under left complement and left least common multiple. Let $a \neq 1$ be an element of M, and let b be in $\mathcal{S} \cup \{1\}$. Then there exists a (unique) greatest right divisor d of a for which $db \in \mathcal{S}$.*

Proof. First, we observe that the set $A = \{z \in M | a = zu$ and $ub \in \mathcal{S}\}$ is non-empty and thus has to have a minimal element with respect to relation "being left divisor" since M is left noetherian. Let c be such a minimal element and assume $a = cd$. Note that by

definition $db \in \mathcal{S}$. We are going to show that d has the desired property. Let $a = uv$ for $u, v \in M$ such that $vb \in \mathcal{S}$. We have to show that v is a right divisor of d.

Observe that vb is the left least common multiple of b and vb, thus v is a left complement of those (cf. Definition 4.1.13) and has to lie in $\mathcal{S}$; similarly we observe that $d \in \mathcal{S}$. Since a is a common left multiple of d and v, they must have the left least common multiple $sd = tv$ and $sd, s, t \in \mathcal{S}$ by assumption. Moreover, there exists $x \in M$ such that $a = x(sd)$. Furthermore, by Lemma 4.1.31, $sdb = tvb$ is the left least common multiple of db and vb, thus lies again in $\mathcal{S}$. So $x \in A$ and, since $a = xsd = cd$, we have $xs = c$ and x is a left divisor of c. By the minimality of c, we have $s = 1$, so $d = tv$. This yields the claim. $\qquad\square$

Notation 4.8.2. *Let M be a locally left Gaussian monoid, $\mathcal{S}$ a generating set closed under left complement and left least common multiple. Let $a \neq 1$ be an element of M, and let b be in $\mathcal{S} \cup \{1\}$. We write $\gamma(a, b)$ for the greatest right divisor d of a for which $db \in \mathcal{S}$. Furthermore, we write $\psi(a, b)$ for the unique element with $a = \psi(a, b)\gamma(a, b)$.*

Let M be a locally left Gaussian monoid and denote by $\overline{B}_* M$ the normalized inhomogeneous bar complex of M. The modules $\overline{B}_n M$ have a canonical basis Ω_n, consisting of all tuples $[x_n | \ldots | x_1]$ satisfying $x_i \neq 1$ for all i. We write Ω_* for the union of all Ω_n. As before, we denote as usual the summands in the differential in the inhomogeneous bar complex by d_i.

Let $\mathcal{S}$ be a generating set for M, closed under left complement and left least common multiple. Define

$$\mathcal{S}_n = \{[x_n | \ldots | x_1] \in \Omega_n \mid \text{For all } 1 \leq k \leq n, x_k \ldots x_1 \in \mathcal{S}\}.$$

The following proposition constructs a noetherian matching on the bar complex (cf. Section 3.2) of a left cancellative, atomic, locally left Gaussian monoid and will be the main part of our proof.

Proposition 4.8.3. *There exists a noetherian matching $\mu \colon \Omega_* \to \Omega_*$ on $\overline{B}_*(M)$ with the property that $x \in \Omega_n$ is a fixed point if and only if $x \in \mathcal{S}_n$.*

Proof. First, we define the **height** of a generator $[x_n | \ldots | x_1] \in \overline{B}_* M$ to be the maximal integer $h \geq 0$ subject to $[x_h | \ldots | x_1] \in \mathcal{S}_h$. If $h = n$, then μ is defined to fix this element. Otherwise, $h + 1 \leq n$ and by definition $x_{h+1} x_h \ldots x_1 \notin \mathcal{S}$. For convenience, set $x_0 = 1$. We now distinguish two cases.

1. If $\gamma(x_{h+1}, x_h \ldots x_1 x_0) = 1$, then we call the cell $[x_n | \ldots | x_1]$ collapsible and set

$$\mu([x_n | \ldots | x_1]) = [x_n | \ldots | x_{h+2} | x_{h+1} x_h | x_{h-1} | \ldots | x_1].$$

   Observe that the new cell has height $h - 1$.

2. If $\gamma(x_{h+1}, x_h \ldots x_1 x_0) \neq 1$, then we call $[x_n | \ldots | x_1]$ redundant and set

$$\begin{aligned} a &= \psi(x_{h+1}, x_h \ldots x_1), \\ d &= \gamma(x_{h+1}, x_h \ldots x_1) \text{ and} \\ \mu([x_n | \ldots | x_1]) &= [x_n | \ldots | x_{h+2} | a | d | x_h | \ldots | x_1]. \end{aligned}$$

Note that $a \neq 1$, because by definition $dx_h \ldots x_1 \in \mathcal{S}$ but

$$adx_h \ldots x_1 = x_{h+1}x_h \ldots x_1 \notin \mathcal{S}.$$

Furthermore, by this argumentation we see that the new cell has height $h + 1$. In particular, if we started with a cell of height 0, we will get into this case since $\gamma(x_1, x_0) = \gamma(x_1, 1)$ is exactly the greatest divisor of x_1 lying in $\mathcal{S}$. The element $\gamma(x_1, 1)$ is non-trivial since $x_1 \neq 1$.

We are now going to show step by step that μ defined in this way is a noetherian matching.

Our first goal is to show that μ is an involution.

Let $\underline{x} = [x_n| \ldots |x_1]$ be redundant of height h. We will first show that

$$\mu(\underline{x}) = [x_n| \ldots |x_{h+2}|a|d|x_h| \ldots |x_1]$$

is collapsible of height $h + 1$.

Set $c = \gamma(a, dx_h \ldots x_1)$. Then $a = yc$ for $y = \psi(a, dx_h \ldots x_1)$, so $x_{h+1} = ad = y(cd)$ and $(cd)x_h \ldots x_1 \in \mathcal{S}$. By definition of d, we have $cd = d$ and $c = 1$. Thus, $\mu(\underline{x})$ is collapsible of height $h + 1$. Hence,

$$\mu^2(\underline{x}) = \mu([x_n| \ldots |x_{h+2}|a|d|x_h| \ldots |x_1]) = [x_n| \ldots |x_{h+2}|ad|x_h| \ldots |x_1] = \underline{x}.$$

Now let $\underline{x} = [x_n| \ldots |x_1]$ be collapsible of height h. We will first show that

$$\mu(\underline{x}) = [x_n| \ldots |x_{h+2}|x_{h+1}x_h|x_{h-1}| \ldots |x_1]$$

is redundant of height $h - 1$.

We have to compute $u = \gamma(x_{h+1}x_h, x_{h-1} \ldots x_1)$. Observe that x_h is a right divisor of $x_{h+1}x_h$ and $x_h(x_{h-1} \ldots x_1) \in \mathcal{S}$ by assumption on $\underline{x}$. So by the definition of u, we have $u = sx_h$ for some $s \in M$, and $x_{h+1}x_h = ru = r(sx_h)$ for $r = \psi(x_{h+1}x_h, x_{h-1} \ldots x_1)$. Thus, $x_{h+1} = rs$ and $u(x_{h-1} \ldots x_1) = s(x_hx_{h-1} \ldots x_1) \in \mathcal{S}$. By definition, s is a right divisor of $\gamma(x_{h+1}, x_h \ldots x_1)$ which is 1 since $\underline{x}$ was collapsible. This implies $s = 1$ and $\gamma(x_{h+1}x_h, x_{h-1} \ldots x_1) = x_h$ by Remark 4.1.30. Since $x_h \neq 1$, this proves that $\mu(\underline{x})$ is redundant of height $h - 1$. Hence,

$$\mu^2(\underline{x}) = \mu([x_n| \ldots |x_{h+2}|x_{h+1}x_h|x_{h-1}| \ldots |x_1]) = \underline{x}.$$

This shows that μ is an involution.

Next, we claim that if $\underline{x}$ is redundant, then $\langle d\mu(\underline{x}), \underline{x} \rangle = \pm 1$.

Let $\underline{x} = [x_n| \ldots |x_1]$ be redundant of height h. Then its partner $\mu(\underline{x})$ is collapsible of height $h + 1$. The claim follows from the observation that $d_{h+1}(\mu(\underline{x})) = \underline{x}$, and that $d_i(\mu(\underline{x})) \neq \underline{x}$ for $i \neq h + 1$. Here is the only place where the left-cancellativity enters. The argument is essentially the same as the proof of (M2) in Lemma 3.4.7.

Last, we are going to check that the matching μ is noetherian.

Let $\underline{x} = [x_n| \ldots |x_1]$ be a redundant cell of height h, and let $\underline{z} \neq \underline{x}$ be redundant with $\underline{x} \vdash \underline{z}$. First, we can assume that $\underline{z}$ is of the form $d_i(\underline{x})$ with $i \neq h$. We want to show that

we can assume $i \notin \{0, n\}$. Recall that $\|m\|$ denotes the maximal number of non-trivial factors $m \in M \setminus \{1\}$ can be expanded into. This number is finite since we assumed M to be atomic. If $d_i([x_n| \ldots |x_1]) = [y_{n-1}| \ldots |y_1]$ and $i \in \{0, n\}$, then

$$\|x_n \ldots x_1\| > \|y_{n-1} \ldots y_1\|.$$

So in a chain with respect to the relation $\vdash$, we may assume that $\|x_n \ldots x_1\|$ is constant and $i \notin \{0, n\}$.

We will prove that now $\underline{z}$ has height at least $h + 1$. For this, let $\underline{y} = \mu(\underline{x}) = [x_n| \ldots |x_{h+2}|a|d|x_h| \ldots |x_1]$ and consider the boundaries $d_i \underline{y}$ for $i \neq h + 1$. We distinguish several cases.

1. $n - 1 \geq i \geq h + 3$: We have $d_i(\underline{y}) = [x_n| \ldots |x_i x_{i-1}| \ldots |x_{h+2}|a|d|x_h| \ldots |x_1]$, which has height $h + 1$ since, as above, $x_k \ldots x_1 \in \mathcal{S}$ for $1 \leq k \leq h$, $d x_h \ldots x_1 \in \mathcal{S}$ and $a d x_h \ldots x_1 \notin \mathcal{S}$. As computed above, $\gamma(a, d x_h \ldots x_1) = 1$, so $d_i(\underline{y})$ is collapsible.

2. $h \geq i \geq 1$: For $i \leq h - 1$ we have $d_i(\underline{y}) = [x_n| \ldots |x_{h+2}|a|d|x_h| \ldots |x_{i+1} x_i| \ldots |x_1]$, and for $i = h$ we have $d_i(\underline{y}) = [x_n| \ldots |x_{h+2}|a|d x_h|x_{h-1}| \ldots |x_1]$. In both cases $d_i(\underline{y})$ has height h, because the product of the first $k \leq h$ entries from the right is $x_m \ldots x_1 \in \mathcal{S}$ for $1 \leq m \leq h$, or $d x_h \ldots x_1 \in \mathcal{S}$, whereas the product of the first $h + 1$ entries from the right gives $a d x_h \ldots x_1 \notin \mathcal{S}$. Computing $\gamma(a, d x_h \ldots x_1) = 1$, we see that $d_i(\underline{y})$ is again collapsible.

3. $i = h + 2$: Here, $d_i(\underline{y}) = [x_n| \ldots |x_{h+3}|x_{h+2}a|d|x_h| \ldots |x_1]$. This cell has height at least $h + 1$, for $d x_h \ldots x_1 \in \mathcal{S}$ and $x_k \ldots x_1 \in \mathcal{S}$ for all $1 \leq k \leq h$. The cell $d_i(\underline{y})$ may or may not be redundant.

Altogether we have shown that if $\underline{z} \neq \underline{x}$ and $\underline{x} \vdash \underline{z}$, then $\underline{z}$ has strictly larger height than $\underline{x}$. Note that the height of a cell is bounded by its dimension. It follows that every chain $\underline{x}_1 \vdash \underline{x}_2 \vdash \ldots$ eventually stabilizes since all $\underline{x}_n$ have the same dimension.

This finishes the proof of the proposition. $\qquad \square$

Applying Theorem 3.2.4 to the matching constructed in the proof of Proposition 4.8.3, we obtain that

$$\theta^\infty \colon (\overline{B}_* M, d) \longrightarrow (\mathcal{S}_*, \theta^\infty \circ d)$$

is a chain homotopy equivalence.

Furthermore, we observe that for $\underline{x} \in \mathcal{S}_*$ we have $\theta^\infty(d(\underline{x})) = d(\underline{x})$ since all boundaries of an essential cell are essential again. (To see this for $d_0[x_n| \ldots |x_1]$, observe that $x_1 \in \mathcal{S}$ and $x_k \ldots x_1 \in \mathcal{S}$, and their least common multiple is $x_k \ldots x_1$. So $x_k \ldots x_2$ lies in $\mathcal{S}$ as a left complement of elements in $\mathcal{S}$.) This yields the following corollary.

Corollary 4.8.4. *Let M be a left cancellative, atomic, locally left Gaussian monoid, $\mathcal{S}$ a generating set closed with respect to left least common multiple and left complement. Then there is a $\mathbb{Z}$-module complex computing the homology of M, with basis $\mathcal{S}_*$ as defined above and differentials given by restriction of the bar differential.*

Next, we want to make this matching into a matching on the classifying space BM for a locally left Gaussian monoid M.

Corollary 4.8.5. *Let M be a left cancellative, atomic, finitely generated, locally left Gaussian monoid, $\mathcal{S}$ a generating set closed with respect to left least common multiple and left complement. (Note that $\mathcal{S}$ is not necessarily finite under these conditions.) Then BM is homotopy equivalent to the subcomplex of all cells given by $\mathcal{S}_*$.*

Proof. We want to use Lemma 3.4.7 first to obtain an acyclic matching on the CW complex BM. This is possible since M is by definition cancellative, has no non-trivial invertible elements by Remark 4.1.30, and a collapsible cell $\underline{x}$ has always height h with $1 \leq h \leq n-1$ and fulfils

$$\mu(\underline{x}) = d_h(\underline{x}).$$

So we obtain out of μ an acyclic matching on BM.

In order to apply Theorem 3.3.8, we have to check that the universal grading for this matching is compact. We will use the Criterion 3.3.9. Define the map

$$
\begin{aligned}
\psi \colon BM^{(*)} &\rightarrow \mathbb{N}, \\
[x_n|\ldots|x_1] &\mapsto \|x_n \ldots x_1\|, \\
[\,] &\mapsto 0.
\end{aligned}
$$

This number is finite since we assumed M to be atomic.

We observe that ψ is a map of posets: Taking boundaries either leaves the value of ψ constant (if it is d_i for $1 \leq i \leq n-1$) or decreases the value (for $i \in \{0, n\}$). Moreover, by the definition of μ, the value of ψ is the same on the elements matched by μ.

Last, we want to show that there are only finitely many elements of $BM^{(*)}$ such that the norm of the product over all entries does not exceed a given value. Let A be any finite generating system for M. Then there are only finitely many elements of M of a fixed A-norm. Surely, the inequality $N_A(m) \leq \|m\|$ holds for any $m \in M \setminus \{1\}$, so there are only finitely many elements $m \in M$ with $\|m\| = k$ for some fixed k. Moreover, if $\|x_n \ldots x_1\| = l$, we know that $\|x_i\| \leq l$ for all $1 \leq i \leq n$ and that $n \leq l$. This implies that the preimage under ψ of any finite subset of $\mathbb{N}$ is finite. So the cells $\psi^{-1}(\{0, 1, \ldots, n\})$ build a finite subcomplex of BM, and we may apply the Criterion 3.3.9. Thus, the compactness condition is also satisfied.

So we obtain a compact, noetherian matching on $BM^{(*)}$, and the essential cells of this matching form a subcomplex. Hence, we can use the Proposition 3.3.10 which implies the claim. $\qquad\square$

Example 4.8.6. *All Artin monoids as in Definition 4.1.7 are atomic, finitely generated, left cancellative and left locally Gaussian. Thus, the corollary above is applicable to them, and we obtain a rather small subcomplex of BM homotopy equivalent to BM. In particular, according to Proposition 4.3.5, the subcomplex of all cells $[x_n|\ldots|x_1]$ with $x_n \ldots x_1$ square-free is homotopy equivalent to BM. In some cases, we can use the last*

corollary to obtain even smaller models for BM, e.g., for the example of Section 4.4, we can use the generating set $\mathcal{E}$ described there to get a finite-dimensional complex homotopy equivalent to BM. Yet, there are already smaller models (with more complicated differentials) known; we will comment on it in more detail in Section 5.2.

Remark 4.8.7. *There are now several chain complexes available for computing the homology of left locally Gaussian monoids. The two complexes already mentioned are the Visy complex associated to the factorability structure as described in Section 4.2 and the complex of Corollary 4.8.4. Dehornoy and Lafont give in [28] a further complex computing the homology of left locally Gaussian monoid; furthermore, they describe a yet different complex computing the homology of a locally Gaussian monoid and compare it in some cases to the Charney-Meier-Whittlesey complex. Moreover, there is a construction by Albenque and Nadeau ([1]) applicable in particular to left locally Gaussian monoids which are atomic and finitely generated. Note that they formulate everything symmetrically with right common multiples.*

We want to compare the different chain complexes. Let M be a left cancellative, atomic, left locally Gaussian monoid and $\mathcal{S}$ a generating system closed under left least common multiples and left complements.

In the Visy complex, we know that the modules $\mathbb{V}_n$ are generated by tuples $[x_n|\ldots|x_1]$ with $x_i \in \mathcal{S}$ and with each pair (x_{i+1}, x_i) being unstable with respect to the factorability structure chosen. In particular, any n-cell of the generalized Charney-Meier-Whittlesey complex as in the Corollary 4.8.4 is contained in the module $\mathbb{V}_n$ of the Visy complex since $x_{i+1}x_i \in \mathcal{S}$ for any two adjacent entries in $[x_n|\ldots|x_1] \in \mathcal{S}_*$. We want to show that the whole complex is a subcomplex of the Visy complex. Thus, we have to evaluate the Visy differential as in Definition 2.2.11. Recall that the differentials can be written as follows:

$$\partial^{\mathbb{V}} = \pi_{n-1} \circ d \circ \sum_{\alpha \in Q_{n-1} \setminus \square_{n-1}} (-1)^{l(\alpha)} f_\alpha \circ i_n,$$

where i_n denotes the inclusion of $\mathbb{V}_n$ in the n-th module of the normalized inhomogeneous bar complex $\overline{B}_n(M)$, d denotes the n-th differential in the same bar complex, and $\pi_{n-1} \colon \overline{B}_n(M) \to \mathbb{V}_n$ denotes the projection. Now observe that any application of f_i to a tuple $[x_n|\ldots|x_1] \in \mathcal{S}_*$ drops the norm, thus the only non-zero summand in the sum above will be given by $\alpha = \varnothing$ (corresponding to no application of f_i). Thus, the Visy differential evaluated at $[x_n|\ldots|x_1] \in \mathcal{S}_*$ is exactly the restriction of the bar differential.

So we know that the generalized Charney-Meier-Whittlesey complex is always a subcomplex of the Visy complex. The comparison with the other complexes is not as easy. The complex obtained from the "order resolution" in [28] has in general much smaller modules: They are given by subsets of any chosen generating system $\mathcal{E}$ of M with certain properties. On the other hand, the differentials are very involved and defined only recursively.

The other complex described in [28], based on the "reversing resolution", is for the case where M is a locally Gaussian monoid and the generating system $\mathcal{S}$ has to be closed under left and right least common multiples and left and right complements. The n-th

module of the complex is generated by all subsets of $\mathcal{S}$ admitting a left common multiple. The differentials are somewhat involved, but explicit. In [28], it is shown that for Garside monoids, this complex contains a complex isomorphic to the Charney-Meier-Whittlesey complex as a subcomplex. The same argument can be used to identify the complex of Corollary 4.8.4 with a subcomplex of the complex obtained by reversing in the case of locally Gaussian monoids.

In [1], there is a complex constructed for an atomic, left-cancellative monoid M with a finite generating system $\mathcal{S}$ with the property that any subset of $\mathcal{S}$ admitting a right common multiple also admits a right least common multiple. Note that atomic, finitely generated right locally Gaussian monoids are special cases of such monoids. Albenque and Nadeau provide a small complex for the homology of such monoids: The generators in degree n are given by the n-element subsets of $\mathcal{S}$ admitting right common multiples. This construction is quite similar to the construction of [56] for Artin monoids.

4.9 Factorable Monoids are not Always Locally Left Gaussian

From the examples seen so far, one could suspect that a non-trivially factorable monoid without non-trivial inverses could have nice divisibility properties, e.g., one could conjecture it is always locally left Gaussian. The following example disproves this conjecture.

Proposition 4.9.1. *Let M be a monoid given by the presentation*

$$\left\langle a, b | ab = ba, a^2 = b^2 \right\rangle^+.$$

Then this monoid is cancellative, has no non-trivial invertible elements and has a factorability structure with respect to the generating system $\{a, b\}$. Yet, there are elements which admit left common multiples but no least left common multiple.

Proof. The first two statements and also the lack of least common multiples are well-known, see e.g. [27], Section II.3.3. Note that any element of the monoid is of the form $a^\varepsilon b^k$ with $\varepsilon \in \{0, 1\}$ and $k \in \mathbb{N}_0$, and this presentation is unique. Note furthermore that the relations have same lengths on both sides, so any product of two elements is geodesic. Define now the factorability structure on M by

$$\eta(a^\varepsilon b^k) = \begin{cases} (a^\varepsilon b^{k-1}, b), & \text{if } k \geq 1 \\ (1, a^\varepsilon), & \text{if } k = 0. \end{cases}$$

We have only to check (F4)+(F5) in order to show that this is a factorability structure with respect to the generating set $\{a, b\}$. We may assume the first entry to have norm > 0. Distinguish the following cases:

Case 1: Consider the pair $(a^\varepsilon b^k, b)$ with $k \geq 1$:

$$
\begin{array}{ccc}
(a^\varepsilon b^k, b) \vdash\!-\!\!\rightarrow (a^\varepsilon b^{k-1}, b, b) & & \\
\top \ \ \Big\downarrow & \top \ \ \Big\downarrow & \\
& (a^\varepsilon b^{k-1}, b^2) \vdash\!-\!\rightarrow (a^\varepsilon b^{k-1}, b, b) & \\
& \top \ \ \Big\downarrow & \\
a^\varepsilon b^{k+1} \vdash\!-\!-\!\rightarrow (a^\varepsilon b^k, b) = \!\!=\!\!= (a^\varepsilon b^k, b)
\end{array}
$$

Case 2: For the pair (a, b), we have

$$
\begin{array}{ccc}
(a, b) \vdash\!-\!\rightarrow (1, a, b) & & \\
\top \ \ \Big\downarrow & \top \ \ \Big\downarrow & \\
& (1, ab) \vdash\!-\!\rightarrow (1, a, b) & \\
& \top \ \ \Big\downarrow & \\
ab \vdash\!-\!-\!\rightarrow (a, b) = \!\!=\!\!= (a, b)
\end{array}
$$

Case 3: Consider now the pair (b^k, a) with $k \geq 1$:

$$
\begin{array}{ccc}
(b^k, a) \vdash\!-\!\rightarrow (b^{k-1}, b, a) & & \\
\top \ \ \Big\downarrow & \top \ \ \Big\downarrow & \\
& (b^{k-1}, ab) \vdash\!-\!\rightarrow (b^{k-1}, a, b) & \\
& \top \ \ \Big\downarrow & \\
ab^k \vdash\!-\!-\!\rightarrow (ab^{k-1}, b) = \!\!=\!\!= (ab^{k-1}, b)
\end{array}
$$

Case 4: For the pair (a, a), we have

$$
\begin{array}{ccc}
(a, a) \vdash\!-\!\rightarrow (1, a, a) & & \\
\top \ \ \Big\downarrow & \top \ \ \Big\downarrow & \\
& (1, b^2) \vdash\!-\!\rightarrow (1, b, b) & \\
& \top \ \ \Big\downarrow & \\
b^2 \vdash\!-\!-\!\rightarrow (b, b) = \!\!=\!\!= (b, b)
\end{array}
$$

Case 5: Last, consider (ab^k, a), $k \geq 1$:

$$(ab^k, a) \vdash \text{-} \blacktriangleright (ab^{k-1}, b, a)$$

$$(ab^{k-1}, ab) \vdash \text{-} \blacktriangleright (ab^{k-1}, a, b)$$

$$b^{k+2} \vdash \text{-} \text{-} \blacktriangleright (b^{k+1}, b) =\!\!=\!\!=\!\!= (b^{k+1}, b)$$

This shows that $\alpha = \beta$ in all cases and, therefore, completes the proof. $\qquad\square$

4.10 Thompson Monoids

In this section, we are going to show that a subfamily of the generalized Thompson monoids introduced by A. Heß consists of locally left Gaussian monoids. This yields also a factorability structure on this monoids, different from the one described in [42]. Thompson's group F itself is a geometric object studied in different contexts (for a short introduction, see e.g. [11], cf. also [42], Section 4.3, [57]; for a detailed introduction and historical background, cf. [13]). It can be defined as the group of piecewise linear self-homeomorphisms of the interval $[0,1]$ with only finitely many breaking points which all have rational coordinates with denominator being a power of 2 and each slope being a (possibly negative) power of 2. This group admits a finite presentation (cf. [13], §3)

$$F \cong \left\langle x_0, x_1 \mid x_0^{-2} x_1 x_0^2 = x_1^{-1} x_0^{-1} x_1 x_0 x_1,\ x_0^{-3} x_1 x_0^3 = x_1^{-1} x_0^{-2} x_1 x_0^2 x_1 \right\rangle.$$

The elements x_0 and x_1 are depicted below.
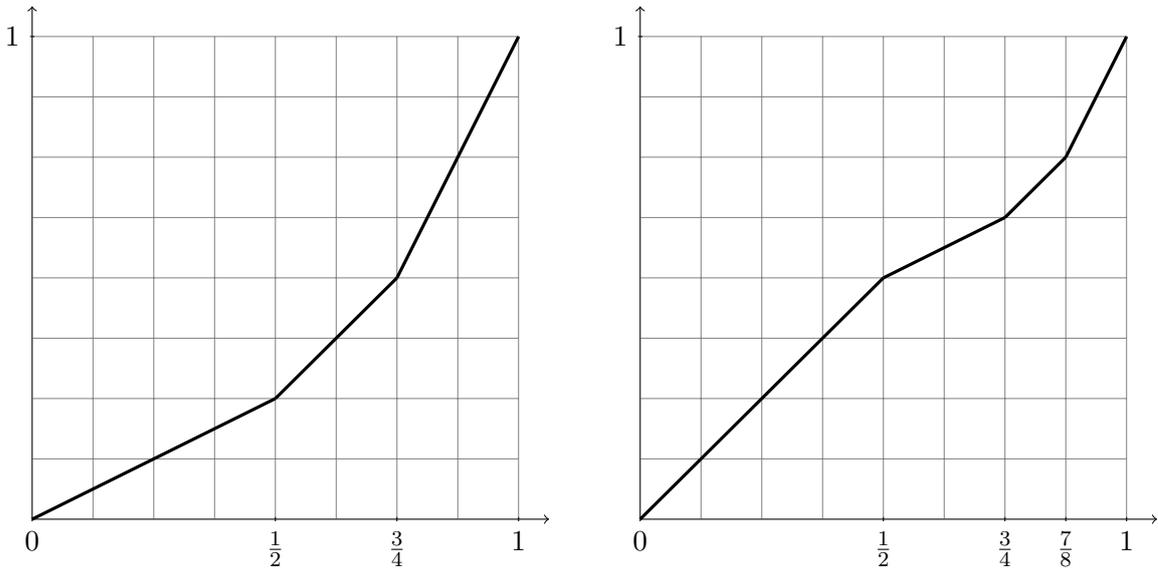
For our purposes, another, infinite presentation is of more interest:

$$F \cong \langle \xi_0, \xi_1, \ldots | \xi_i \xi_k = \xi_k \xi_{i+1}, \ i > k \rangle.$$

Here, ξ_0 and ξ_1 are the same functions as x_0 and x_1 above; the other ξ_i may be defined as $\xi_i = \xi_0^{-1} \xi_{i-1} \xi_0$. One can also say that each ξ_i is identity on $[0, \frac{1}{2}]$ and a rescaled version of ξ_{i-1} on $[\frac{1}{2}, 1]$ (cf. [11]).

We will consider the following class of monoids first defined by A. Heß.

Definition 4.10.1. *([42]) We define **(generalized) Thompson monoids** by*

$$t_{p,q}^{(m)} = \langle \xi_0, \xi_1, \ldots, \xi_{m-1} | \xi_{i-p} \xi_k = \xi_k \xi_i, \ i - k \geq q, i \geq p \rangle$$

for $p, q \geq 0$. We also allow $m = \infty$.

Remark 4.10.2.   *1. For $q = p + 1 = 2$ and $m = \infty$, the associated group of the monoid $t_{1,2}^\infty$ is the "classical" Thompson group F. As shown by A. Heß ([42], Section 4.3), this monoid satisfies the Ore condition and hence embeds into the Thompson group F.*

 2. In [42], the Thompson monoid $t_{p,q}^{(m)}$ is defined to have generators $\xi_1, \ldots, \xi_m$. This index shift is not an issue; the monoids $t_{p,q}^{(m)}$ in our definition are isomorphic to those of [42].

We are going to show:

Proposition 4.10.3. *Thompson monoids $t_{p,q}^{(m)}$ are locally left Gaussian for the parameters $q \geq p \geq 1$.*

Proof. We are going to check that the assumptions of the Theorem 4.1.28 hold for the presentation of $t_{p,q}^{(m)}$ from Definition 4.10.1. The presentation is obviously positive. Furthermore, each pair (i, k) of indices between 0 and $m - 1$ belongs to at most one relation of the form $u\xi_k = v\xi_i$ since exactly one of the inequalities $i - k \geq q$, $k - i \geq q$ or $-q < i - k < q$ is satisfied. Thus, the presentation is complemented. Furthermore, the first condition of Theorem 4.1.28 is fulfilled since the presentation is homogeneous.

It remains to check the second condition. This will involve a lengthy case distinction. Note that whenever $i - k \geq q$, we have in particular $i \geq k + q \geq q \geq p$ by assumption. So the condition $i \geq p$ from the defining relations of $t_{p,q}^{(m)}$ is always satisfied if $i - k \geq q$ since we assumed $q \geq p$. Let $\alpha = \xi_a$, $\beta = \xi_b$, $\gamma = \xi_c$ be three different generators. Note that for $0 \leq r, s \leq m - 1$ with $r - s \geq q$, we have the relation $\xi_{r-p} \xi_s = \xi_s \xi_r$ and thus $\xi_r / \xi_s = \xi_{r-p}$ and $\xi_s / \xi_r = \xi_s$. Note also that the left-hand side of the condition to check can only be defined if $|a - b| \geq q$ and $|b - c| \geq q$ (it may though even be undefined if these inequalities hold, as we will see in some cases). The same way, the right-hand side can only be defined if $|a - c| \geq q$ and $|b - c| \geq q$. Here is the list of the considered cases:

Case 1: If $-\mathbf{q} < \mathbf{b} - \mathbf{c} < \mathbf{q}$, then both sides are undefined. We may assume that $|b - c| \geq q$ in the following cases.

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Case 2: The inequality $\mathbf{b} - \mathbf{c} \geq \mathbf{q}$ implies $\xi_c/\xi_b = \xi_c$ and $\xi_b/\xi_c = \xi_{b-p}$. We proceed by distinguishing several subcases:

Case 2.1: The additional inequality $\mathbf{a} - \mathbf{b} \geq \mathbf{q}$ implies $a - c \geq 2q$. So the left-hand side is ξ_{a-p}/ξ_c. Now $(a - p) - c = a - c - p \geq 2q - p \geq q$, so $\xi_{a-p}/\xi_c = \xi_{a-2p}$. On the other hand, we have $\xi_a/\xi_c = \xi_{a-p}$, and the right-hand side is given by ξ_{a-p}/ξ_{b-p}. To compute this, note that $(a - p) - (b - p) = a - b \geq q$, so $\xi_{a-p}/\xi_{b-p} = \xi_{a-2p}$. Thus, both sides coincide.

Case 2.2: Now assume $-\mathbf{q} < \mathbf{a} - \mathbf{b} < \mathbf{q}$. As we mentioned before, the left-hand side is undefined. Furthermore, the inequality implies $-q < 0 < a - c$.

Case 2.2(a): If $\mathbf{a} - \mathbf{c} < \mathbf{q}$, then the right-hand side is also undefined.

Case 2.2(b): If $\mathbf{a} - \mathbf{c} \geq \mathbf{q}$, we have as the right-hand side ξ_{a-p}/ξ_{b-p}. Now

$$-q < (a - p) - (b - p) = a - b < q,$$

thus the right-hand side is undefined as well.

Case 2.3: Now let $\mathbf{b} - \mathbf{a} \geq \mathbf{q}$. Thus $\xi_a/\xi_b = \xi_a$, and the left-hand side is ξ_a/ξ_c. We have again to consider several subcases:

Case 2.3(a): If $\mathbf{a} - \mathbf{c} \geq \mathbf{q}$, then we have $\xi_a/\xi_c = \xi_{a-p}$ on the left-hand side. On the right-hand side, we obtain ξ_{a-p}/ξ_{b-p} and since

$$(b - p) - (a - p) = b - a \geq q,$$

we have $\xi_{a-p}/\xi_{b-p} = \xi_{a-p}$ on the right-hand side.

Case 2.3(b): Consider now $-\mathbf{q} < \mathbf{a} - \mathbf{c} < \mathbf{q}$. The right-hand side is obviously undefined. Yet, the expression ξ_a/ξ_c on the left-hand side is also undefined.

Case 2.3(c): Last, we deal with the case $\mathbf{c} - \mathbf{a} \geq \mathbf{q}$. The left-hand side is now equal to ξ_a. On the right-hand side, we have ξ_a/ξ_{b-p}. Note that

$$b - a = (b - c) + (c - a) \geq 2q,$$

thus $(b - p) - a = b - a - p \geq 2q - p \geq q$, so the right-hand side amounts to ξ_a.

Case 3: Now we assume $\mathbf{c} - \mathbf{b} \geq \mathbf{q}$ which implies $\xi_c/\xi_b = \xi_{c-p}$ and $\xi_b/\xi_c = \xi_b$. We have to consider several subcases again.

Case 3.1: We start with $\mathbf{b} - \mathbf{a} \geq \mathbf{q}$. Thus, we have $\xi_a/\xi_b = \xi_a$ and $c - a \geq 2q$. The left-hand side is ξ_a/ξ_{c-p} and since $(c - p) - a = (c - a) - p \geq 2q - p \geq q$, we see that the left-hand side equals ξ_a. For the right-hand side, we first obtain ξ_a/ξ_b, which is ξ_a.

Case 3.2: Assume next the inequality $-\mathbf{q} < \mathbf{b} - \mathbf{a} < \mathbf{q}$ holds, thus the left-hand side is undefined. We observe furthermore $c - a = (c - b) + (b - a) > 0$. This allows two different relation types:

Case 3.2(a): If $\mathbf{c} - \mathbf{a} < \mathbf{q}$, then the right-hand side is also immediately undefined.

Case 3.2(b): If $\mathbf{c} - \mathbf{a} \geq \mathbf{q}$, then the right-hand side equals ξ_a / ξ_b which is in turn undefined.

Case 3.3: Now let $\mathbf{a} - \mathbf{b} \geq \mathbf{q}$. On the left-hand side, the expression simplifies to ξ_{a-p}/ξ_{c-p}. For a and c, we have to deal with three cases again:

Case 3.3(a): For $\mathbf{a} - \mathbf{c} \geq \mathbf{q}$, note that we even have $a - b = (a - c) + (c - b) \geq 2q$. Then $\xi_{a-p}/\xi_{c-p} = \xi_{a-2p}$ since $(a - p) - (c - p) = a - c \geq q$. For the right-hand side, we have ξ_{a-p}/ξ_b which equals ξ_{a-2p} since

$$(a - p) - b = (a - b) - p \geq 2q - p \geq q.$$

Case 3.3(b): Consider the case $-\mathbf{q} < \mathbf{a} - \mathbf{c} < \mathbf{q}$. The right-hand side is undefined. The left-hand side is undefined as well since $-q < (a - p) - (c - p) = a - c < q$.

Case 3.3(c): Last, we assume $\mathbf{c} - \mathbf{a} \geq \mathbf{q}$. The left-hand side is ξ_{a-p}/ξ_{c-p} and since $(c - p) - (a - p) = c - a \geq q$, we see that the left-hand side equals ξ_{a-p}. On the right-hand side, we get $\xi_a/\xi_b = \xi_{a-p}$.

This covers all possible combinations of three different generators (by which we mean elements of X$=\{\xi_0, \xi_1, \ldots, \xi_{m-1}\}$). Thus, Thompson monoids with $q \geq p \geq 1$ are locally left Gaussian. $\qquad\square$

Now we are going to investigate a generating set for $t_{p,q}^{(m)}$ which is closed under left least common multiples and left complement. For the rest of the section, fix $q \geq p + 1 \geq 2$. We make this further restriction since we need the following fact proven by A. Heß.

Lemma 4.10.4 ([42], Section 4.3)**.** *The monoids $t_{p,q}^{(m)}$ are left cancellative for $p < q$.*

We want to show:

Proposition 4.10.5. *The set*

$$\{\xi_{i_1} \ldots \xi_{i_k} | i_{j+1} - i_j \geq q \text{ for all } 1 \leq j \leq k - 1\} \subset t_{p,q}^{(m)}$$

is closed under left least common multiples and left complements.

First, we make several observations about left least common multiples in these monoids.

Remark 4.10.6. *By Remark 4.1.29, we know that the left complement of ξ_a and ξ_b is represented by ξ_a/ξ_b.*

In particular, we know that the left least common multiple of two distinct generators ξ_a and ξ_b exists if and only if $|a - b| \geq q$, and is then $\xi_a \xi_b$ if $b - a \geq q$ and $\xi_b \xi_a$ if $a - b \geq q$.

Definition 4.10.7. *Let $\xi_{i_1} \ldots \xi_{i_k}$ be a word in $\xi_0, \ldots, \xi_{m-1}$. We call this word **ascending** if for all $1 \leq l \leq k - 1$, we have $i_{l+1} - i_l \geq q$.*

Remark 4.10.8. *Note that the ascending words are exactly the totally unstable words in the sense of [42], Section 4.3.*

We will show the following divisibility property for elements represented by ascending words.

Lemma 4.10.9. *Let $\xi_{i_1} \ldots \xi_{i_k}$ be an ascending word representing $x \in t_{p,q}^{(m)}$. Then for any $1 \leq l \leq k$, there is an ascending word $\xi_{j_1} \ldots \xi_{j_{k-1}}$ representing $y \in t_{p,q}^{(m)}$ such that $x = y\xi_{i_l}$.*

Proof. The statement is clear for $l = k$. If $l < k$, consider the subword $\xi_{i_{l-1}}\xi_{i_l}\xi_{i_{l+1}}$. By definition, we know that $\xi_{i_l}\xi_{i_{l+1}} = \xi_{i_{l+1}-p}\xi_{i_l}$, and

$$(i_{l+1} - p) - i_{l-1} = (i_{l+1} - i_l) + (i_l - i_{l-1}) - p \geq 2q - p \geq q,$$

so the word $\xi_{i_1} \ldots \xi_{i_{l-1}}\xi_{i_{l+1}-p}$ to the right of ξ_{i_l} is ascending. (The latter consideration is superfluous if $l = 1$.) Furthermore, if ξ_{i_l} is not on the very right after this manipulation, we can continue moving it to the right since $i_{l+2} - i_l \geq q$. The resulting word for y is given by $\xi_{i_1}\xi_{i_2} \ldots \xi_{i_{l-1}}\xi_{i_{l+1}-p} \ldots \xi_{i_k-p}$, which is obviously ascending. This yields the claim. $\square$

We compute the left least common multiples for elements represented by ascending words. In the first step, we compute the left least common multiple of such an element with a generator ξ_l.

Lemma 4.10.10. *Let $\xi_{i_1} \ldots \xi_{i_k}$ be an ascending word representing $x \in t_{p,q}^{(m)}$. Then ξ_l right-divides x if and only if $l \in \{i_1, \ldots, i_k\}$. Furthermore, x and ξ_l have a left common multiple if and only if $\{r \mid -q < r - l < q\} \cap \{i_1, \ldots, i_k\} \subset \{l\}$. If the left least common multiple of x and ξ_l exists, it and also the left complements x/ξ_l and ξ_l/x can be represented by ascending words.*

Moreover, if $\{r \mid -q < r - l < q\} \cap \{i_1, \ldots, i_k\} = \varnothing$, consider the unique index $0 \leq s \leq k$ such that $i_{s+1} - l \geq q$ and $l - i_s \geq q$ (if $s = 0$ or $s = k$, the inequality which doesn't make sense is omitted). Then the word $\xi_{i_1} \ldots \xi_{i_s}\xi_l\xi_{i_{s+1}} \ldots \xi_{i_k}$ represents the left least common multiple of x and ξ_l.

Proof. Let l be an element of $\{0, \ldots, m-1\}$.

From Lemma 4.10.9, we immediately see that ξ_l right-divides x if $l \in \{i_1, \ldots, i_k\}$.

Suppose now that $\{i_1, \ldots, i_k\}$ contains an element r of $\{r \mid -q < r - l < q\}$ which is different from l. Then by Lemma 4.10.9, we know that x is a left multiple of ξ_r. By Remark 4.10.6, we know that x cannot have a left common multiple with ξ_l. For the rest of the proof, we thus may assume that $\{r \mid -q < r - l < q\} \cap \{i_1, \ldots, i_k\} \subset \{l\}$.

We now show by induction on k that $l \notin \{i_1, \ldots, i_k\}$ implies that x is not right-divisible by ξ_l. For $k = 1$, this is clear since any non-trivial multiple of ξ_l has to have word length larger than 1 with respect to the ξ_i's. Assume now we have shown the claim for $k - 1$, and assume ξ_l divides x. Then x is a left common multiple of ξ_{i_k} and ξ_l. We consider two cases: First, assume $i_k - l \geq q$. Then $\xi_{i_k-p}\xi_l = \xi_l\xi_{i_k}$ is by Remark 4.10.6 a left

least common multiple of ξ_{i_k} and ξ_l. Thus, there is a $z \in t_{p,q}^{(m)}$ such that $x = z\xi_l\xi_{i_k}$. This, in turn, implies by right cancellation that ξ_l right-divides $\xi_{i_1} \ldots \xi_{i_{k-1}}$ which is again represented by an ascending word. So we obtain a contradiction by induction hypothesis.

Now assume $l - i_k \geq q$. (Note that we already excluded $-q < l - i_k < q$.) Then $\xi_{l-p}\xi_{i_k} = \xi_{i_k}\xi_l$ is the left least common multiple of ξ_l and ξ_{i_k}, and there is a $z \in t_{p,q}^{(m)}$ such that $x = z\xi_{l-p}\xi_{i_k}$. Cancelling on the right again, we see that ξ_{l-p} must right-divide $\xi_{i_1} \ldots \xi_{i_{k-1}}$. By induction hypothesis, $l - p \in \{i_1, \ldots, i_{k-1}\}$. But since

$$-q < (l-p) - l = -p < q,$$

this is a contradiction to $\{r \mid -q < r - l < q\} \cap \{i_1, \ldots, i_k\} \subset \{l\}$. So we have proven the first assertion.

Now if $\{r \mid -q < r - l < q\} \cap \{i_1, \ldots, i_k\} = \varnothing$, there is a unique index $0 \leq s \leq k$ such that $i_{s+1} - l \geq q$ and $l - i_s \geq q$ (if $s = 0$ or $s = k$, the inequality which doesn't make sense is omitted). Consider the word $\xi_{i_1} \ldots \xi_{i_s}\xi_l\xi_{i_{s+1}} \ldots \xi_{i_k}$ representing $u \in t_{p,q}^{(m)}$. Then this word is by definition ascending and represents an element right-divisible by ξ_l by Lemma 4.10.9. This lemma also shows that the element y with $u = y\xi_l$ can be represented by an ascending word. Furthermore, u is also divisible by x:

$$
\begin{aligned}
u &= \xi_{i_1} \ldots \xi_{i_s}\xi_l\xi_{i_{s+1}} \ldots \xi_{i_k} \\
&= \xi_{i_1} \ldots \xi_{i_{s-1}}\xi_{l-p}\xi_{i_s}\xi_{i_{s+1}} \ldots \xi_{i_k} \\
&= \ldots \\
&= \xi_{l-sp}\xi_{i_1}\xi_{i_2} \ldots \xi_{i_k} = \xi_{l-sp}x.
\end{aligned}
$$

(Note that $l - sp > 0$ since, by definition of s and of an ascending sequence, we have $l \geq i_s + q \geq i_{s-1} + 2q \geq \ldots \geq i_1 + sq > sp$.)

Since we have already shown that ξ_l does not right-divide x, the element u is also a left least common multiple of x and ξ_l for word length reasons: since u provides a common multiple of x and ξ_l, we know that there is the left least common multiple w for x and ξ_l. Since the relations in ξ_i's have same length on both sides, any word representing an element of a Thompson monoid has the same word length in the ξ's. So the word length of w must be at least $k + 1$ since $w \neq x$ as shown before. On the other hand, w is a divisor of u and the word length of u is also $k + 1$. This yields $u = w$. Since ξ_{l-sp} is an ascending word, we are done. $\qquad\square$

Now we can proceed to the general case.

Lemma 4.10.11. *For any two elements in $t_{p,q}^{(m)}$ represented by ascending words, their left least common multiple can be also represented by an ascending word if it exists.*

Proof. Let $\xi_{i_1} \ldots \xi_{i_k}$ and $\xi_{j_1} \ldots \xi_{j_l}$ be ascending words representing the elements x and y of $t_{p,q}^{(m)}$. First, we observe that if there is an $a \in \{i_1, \ldots, i_k\}$ and a $b \in \{j_1, \ldots, j_l\}$ with $-q < a - b < q$ and $a \neq b$, then there is no left common multiple for x and y. Indeed,

since x is then a left multiple of ξ_a and y a left multiple of ξ_b, a left common multiple for x and y would be a left common multiple for ξ_a and ξ_b, which is impossible by Remark 4.10.6.

Now assume there are no such a, b in the index sets. We will show the statement by induction on l. For $l = 1$, the claim is just Lemma 4.10.10. For the induction step, note that taking left least common multiple is associative and, using the proof of Lemma 4.10.10 again, l-lcm$(\xi_{j_1} \ldots \xi_{j_{l-1}}, \xi_{j_l}) = \xi_{j_1} \ldots \xi_{j_l}$. Thus we have

$$\text{l-lcm}(\xi_{i_1} \ldots \xi_{i_k}, \xi_{j_1} \ldots \xi_{j_l}) = \text{l-lcm}(\xi_{i_1} \ldots \xi_{i_k}, \text{l-lcm}(\xi_{j_1} \ldots \xi_{j_{l-1}}, \xi_{j_l}))$$
$$= \text{l-lcm}(\text{lcm}(\xi_{i_1} \ldots \xi_{i_k}, \xi_{j_1} \ldots \xi_{j_{l-1}}), \xi_{j_l}).$$

But l-lcm$(\xi_{i_1} \ldots \xi_{i_k}, \xi_{j_1} \ldots \xi_{j_{l-1}})$ can be represented by an ascending word by the induction hypothesis, and thus l-lcm(l-lcm$(\xi_{i_1} \ldots \xi_{i_k}, \xi_{j_1} \ldots \xi_{j_{l-1}}), \xi_{j_l})$ can be represented by an ascending word by Lemma 4.10.10. $\qquad\square$

We are ready to prove the Proposition 4.10.5.

Proof. (of the Proposition 4.10.5)

We have already shown in the last lemma that the set in the statement of the proposition is closed under left least common multiple. Now let x and y be two elements of $t_{p,q}^{(m)}$ which can be represented by ascending words. Let z be their left least common multiple. Then there is a $w \in t_{p,q}^{(m)}$ with $z = wx$, and we want to show that w can be represented by an ascending word. We will show inductively that for any $a = bc$ in $t_{p,q}^{(m)}$ with a, c represented by ascending words, b can be also represented by an ascending word. For word length of c equal to 1, the claim is given by Lemma 4.10.10. For the induction step, assume c is represented by an ascending word $\xi_{i_1} \ldots \xi_{i_k}$ with $k > 1$. In particular, ξ_{i_k} right-divides a, so there is an ascending word a' such that $a = a'\xi_{i_k}$ and, by left cancellation due to Lemma 4.10.4, we obtain $a' = b\xi_{i_1} \ldots \xi_{i_{k-1}}$. Thus, by induction hypothesis, b can be represented by an ascending word. $\qquad\square$

From the Propositions 4.10.3, 4.10.5 and 4.2.5, we get:

Corollary 4.10.12. *The monoid $t_{p,q}^{(m)}$ is factorable with respect to the generating system* $\{\xi_{i_1} \ldots \xi_{i_k} | i_{j+1} - i_j \geq q \text{ for all } 1 \leq j \leq k - 1\}$.

5 Homology of Artin Groups and Monoids

In this chapter, we are going to give an overview of different attempts to determine the homology of Artin monoids. Their homology is linked to the homology of Artin groups by the $K(\pi, 1)$-conjecture. We will treat it in Section 5.3.

5.1 Brieskorn-Saito Normal Forms

In this section, we present a rather small complex computing the homology of Artin monoids. It arises from a noetherian matching on the bar complex, induced by the Brieskorn-Saito normal form (cf. [10]). First, we will recall some facts about these normal forms. Then we imitate the proof of Brown that complete rewriting systems yield noetherian matchings on the normalized bar complex (cf. [22], also [42] for the proof that factorability structures yield noetherian matchings). It is natural to conjecture that the Brieskorn-Saito normal form comes from a complete rewriting system, yet it seems complicated to write it down explicitly if it exists.

Let $M := M(S)$ be an Artin monoid with Artin-Coxeter generating set S. Let $W(S)$ be the corresponding Coxeter group. For $I \subset S$, let $W(I)$ be the Coxeter group given by the restriction of the Coxeter matrix to I. (It is well-known that $W(I)$ is the subgroup of $W(S)$ generated by I). Set

$$S^f = \{I \subset S | W(I) \text{ is finite}\}.$$

Furthermore, set $\mathcal{D} = \{\Delta_I := \text{l-lcm}(I) | \varnothing \neq I \in S^f\}$. Note that these least common multiples exist, which follows from the proposition below.

Observe that in [10], Brieskorn-Saito normal forms are left normal forms. We will formulate everything symmetrically since we always work with right normal forms; it is clear that symmetric statements always hold: We use the anti-automorphism of $M(S)$ (similarly for $W(S)$ or $G(S)$) defined by $s_1 s_2 \ldots s_k \mapsto s_k \ldots s_2 s_1$ for any representative $s_1 s_2 \ldots s_k$ with $s_i \in S$. This anti-automorphism makes left least common multiples out of right least common multiples, left divisors out of right divisors and so on.

Proposition 5.1.1 ([10], 5.6). *In an Artin monoid, a fundamental element (i.e. left least common multiple of all Artin-Coxeter generators) exists if and only if the corresponding Coxeter group is finite.*

Note furthermore that $S \subset \mathcal{D}$ since $\{a\} \in S^f$ for all $a \in S$. In particular, $\mathcal{D}$ is a generating set for M.

For $x \in M$, let $I(x) := \{a \in S | \exists y \in M \colon x = ya\}$ be the set of letters in S with which a word for x may start (on the right). Note that $I(\Delta_J) = J$ for any subset $J \subset S$ admitting the left least common multiple (cf. e.g. [56], Part II, §5). The Brieskorn-Saito normal form is given by the following theorem.

Theorem 5.1.2 ([10], §6). *For any $w \in M$, there are unique non-empty subsets*

$$I_k, \ldots, I_1 \subset S$$

such that

$$w = \Delta_{I_k} \Delta_{I_{k-1}} \dots \Delta_{I_2} \Delta_{I_1}$$

and $I(\Delta_{I_k} \dots \Delta_{I_j}) = I_j$ for $1 \leq j \leq k$.

Remark 5.1.3. *This set of normal forms does not consist only of words of minimal length with respect to $\mathcal{D}$. For example, in the monoid $M_{4,4}$, we can see that for $x = babac = \Delta_{\{a,b\}}\Delta_c$, the normal form of x is given by $\Delta_b\Delta_a\Delta_b\Delta_{\{a,c\}}$. So the normal form is not geodesic.*

We now define a noetherian matching on the bar complex of an Artin monoid M, relying on the Brieskorn-Saito normal form. As already mentioned, we will mimic the proof of Brown like in [22], §7.3, and the proof of Heß like in [42], §1.2.4 and §3.1. Only for the noetherian property, one needs some new arguments. In this section, we always mean by "normal form" the Brieskorn-Saito normal form.

We denote the graded set of standard basis elements of the bar complex by Ω_*. Consider a cell $[x_n|\dots|x_1] \in \Omega_*$ of the bar complex of the monoid. We denote by $w_i \in \mathcal{D}^*$ the normal form of x_i.

Definition 5.1.4. *We say that a cell $[x_n|\dots|x_1]$ is **essential** if the following three conditions hold:*

(a) $x_1 \in \mathcal{D}$,

(b) The word $w_{i+1}w_i$ is not the normal form of $x_{i+1}x_i$ for every $1 \leq i \leq n-1$,

(c) For every $1 \leq i \leq n-1$, any proper (right) prefix of $w_{i+1}w_i$ is a normal form.

*For any cell $[x_n|\dots|x_1]$, we define its **height** by*

$$\mathrm{ht}([x_n|\dots|x_1]) = \max\{j \mid [x_j|\dots|x_1] \text{ is essential}\}.$$

*If $x_1 \notin \mathcal{D}$, set $\mathrm{ht}([x_n|\dots|x_1]) = 0$. Clearly, an n-cell is essential iff its height is n. We partition the non-essential cells into **redundant** and **collapsible** according to the following rules.*

- *If $\mathrm{ht}([x_n|\dots|x_1]) = 0$, call the cell redundant.*

- *If $\mathrm{ht}([x_n|\dots|x_1]) = h > 0$ and $w_{h+1}w_h$ is the normal form of $x_{h+1}x_h$, call the cell collapsible.*

- *If $\mathrm{ht}([x_n|\dots|x_1]) = h > 0$ and $w_{h+1}w_h$ is not the normal form of $x_{h+1}x_h$, but also some proper prefix of $w_{h+1}w_h$ is not a normal form, call the cell redundant.*

Note that any cell is now either essential or redundant or collapsible.

We are now in the position to define the matching $\mu\colon \Omega_* \to \Omega_*$. We define μ to be the identity on the essential cells. If $[x_n|\ldots|x_1]$ is collapsible of height h (observe $0 < h < n$), we define

$$\mu([x_n|\ldots|x_1]) = [x_n|\ldots|x_{h+2}|x_{h+1}x_h|x_{h-1}|\ldots|x_1].$$

If $[x_n|\ldots|x_1]$ is redundant of height 0 and $I = I(x_1)$, there is by definition a unique $y \in M \setminus \{1\}$ with $x_1 = y\Delta_I$. Then define

$$\mu([x_n|\ldots|x_1]) = [x_n|\ldots|x_2|y|\Delta_I].$$

Last, if $[x_n|\ldots|x_1]$ is redundant of height $h > 0$, i.e., some proper prefix of $w_{h+1}w_h$ is not a normal form, let $w_{h+1} = ab \in \mathcal{D}^*$ with b minimal such that bw_h is not a normal form. By definition, $a \neq 1$, and $b \neq 1$ since w_h is a normal form. Set in this case

$$\mu([x_n|\ldots|x_1]) = [x_n|\ldots|x_{h+2}|a|b|x_h|x_{h-1}|\ldots|x_1].$$

We are going to prove the following proposition.

Proposition 5.1.5. *For any Artin monoid M, the map $\mu\colon \Omega_* \to \Omega_*$ defined as above gives a noetherian, $\mathbb{Z}$-compatible matching on the bar complex.*

We begin by some preliminary considerations.

Lemma 5.1.6. *If $\Delta_{J_k}\Delta_{J_{k-1}}\ldots\Delta_{J_2}\Delta_{J_1}$ is a normal form, then so are $\Delta_{J_k}\Delta_{J_{k-1}}\ldots\Delta_{J_2}$ and $\Delta_{J_{k-1}}\ldots\Delta_{J_1}$.*

Proof. The first claim follows directly from the definition.

Observe that for any product $xy \in M$, we have $I(y) \subset I(xy)$. Furthermore, $I(\Delta_J) = J$. (cf. [10]). For the second part, we proceed by induction on k. For $k = 1$ and $k = 2$, the claim is clear. Assume we have proven the claim for $k - 1$. For k, we have

$$J_1 \subset I(\Delta_{J_{k-1}}\ldots\Delta_{J_1}) \subset I(\Delta_{J_k}\Delta_{J_{k-1}}\ldots\Delta_{J_2}\Delta_{J_1}) = J_1$$

thus the last generator in the normal form of $\Delta_{J_{k-1}}\ldots\Delta_{J_1}$ is Δ_{J_1}. So we have to show that $\Delta_{J_{k-1}}\ldots\Delta_{J_2}$ is a normal form. This follows from the first statement of the Lemma and from the induction hypothesis. $\square$

For the proof of noetherianity, the following partial order will be useful.

Definition 5.1.7. *We define a partial order on the tuples of subsets of S as follows: set $(A_r, A_{r-1}, \ldots, A_1) < (B_s, B_{s-1}, \ldots, B_1)$ if, for some $k \leq \max\{r, s\}$, $A_i = B_i$ for $1 \leq i < k$ and $A_k \subsetneq B_k$ holds. (If $i > r$, set $A_i = \varnothing$, similar for B.)*

Definition 5.1.8. *For any $x \in M$ with normal form $x = \Delta_{I_k}\Delta_{I_{k-1}}\ldots\Delta_{I_2}\Delta_{I_1}$, set $\widetilde{I}(x)$ to be the tuple $(I_k, I_{k-1}, \ldots, I_1)$ of subsets of S.*

For a basis element of the bar complex $[x_n|\ldots|x_1]$, let $\widetilde{I}([x_n|\ldots|x_1])$ be the tuple of subsets of S obtained by concatenation of the tuples $\widetilde{I}(x_n), \ldots, \widetilde{I}(x_1)$.

Lemma 5.1.9. *1. If $x = \Delta_{J_s} \ldots \Delta_{J_1}$ is any expression in $\mathcal{D}$ representing x with normal form $x = \Delta_{I_k}\Delta_{I_{k-1}}\ldots\Delta_{I_2}\Delta_{I_1}$, then $(J_s, \ldots, J_1) \leq (I_k, \ldots, I_1)$, and the equality holds if and only if $\Delta_{J_s} \ldots \Delta_{J_1}$ is the normal form.*

2. For any $x, y \in M$, we have $\widetilde{I}(xy) \geq (\widetilde{I}(x), \widetilde{I}(y))$, and the equality holds if and only if the normal form of xy is the product of normal forms of x and y.

Proof. The second part follows obviously from the first one by considering the concatenation of normal forms of x and y as an expression for xy.

For the first part, we proceed by induction on s. If $x = \Delta_J$, this is already the normal form, so the claim is proven for $s = 1$. Now assume we already have proven the claim for $s - 1$. We have $J_1 \subset I(x) = I_1$, thus either $J_1 \subsetneq I_1$ and we are done, or $J_1 = I_1$. In the second case, $\Delta_{J_s} \ldots \Delta_{J_2}$ has the normal form $\Delta_{I_k}\Delta_{I_{k-1}}\ldots\Delta_{I_2}$ and, by induction hypothesis, $(J_s, \ldots, J_2) \leq (I_k, \ldots, I_2)$. Since $J_1 = I_1$, it follows also $(J_s, \ldots, J_2, J_1) \leq (I_k, \ldots, I_2, I_1)$. The equality case is clear. This completes the proof. $\square$

We are now ready to prove the last proposition.

Proof. [of Proposition 5.1.5] We first prove that μ is an involution. This is clear for essential cells. Now let $[x_n| \ldots |x_1]$ be a collapsible cell of height h. Then

$$\mu([x_n| \ldots |x_1]) = [x_n| \ldots |x_{h+2}|x_{h+1}x_h|x_{h-1}| \ldots |x_1].$$

Note that $1 \leq h < n$.

Consider the case $h = 1$. Then $x_2x_1 \notin \mathcal{D}$ and the image cell is redundant of height 0. Since the cell we started with was collapsible of height 1, we know that $x_1 \in \mathcal{D}$ and the normal form of x_2x_1 is the product of normal forms of the factors. Thus, x_1 is the last generator in the normal form of x_2x_1 and, by definition, $\mu^2([x_n| \ldots |x_1]) = [x_n| \ldots |x_1]$. (Here, we for simplicity do not distinguish between x_1 and its normal form w_1 since, by definition, both consist of the same single element of $\mathcal{D}$.)

Now assume we have a collapsible cell of height $h > 1$. Then $[x_{h-1}| \ldots |x_1]$ is by definition essential, so the height of $\mu([x_n| \ldots |x_1])$ is at least $h - 1$. We first want to show that $[x_{h+1}x_h|x_{h-1}| \ldots |x_1]$ is not essential which will imply that the height of $\mu([x_n| \ldots |x_1])$ is exactly $h - 1$. Recall we denote by w_i the normal form of x_i, and that $w_{h+1}w_h$ is the normal form of $x_{h+1}x_h$ by assumption. We know that w_hw_{h-1} is not the normal form for x_hx_{h-1}, thus by Lemma 5.1.6, $w_{h+1}w_hw_{h-1}$ is also not a normal form. This implies that $[x_n| \ldots |x_{h+2}|x_{h+1}x_h|x_{h-1}| \ldots |x_1]$ is redundant of height $h - 1$, and that w_h is the minimal prefix b of $w_{h+1}w_h$ such that bw_{h-1} is not the normal form (minimality follows from condition (c) for the original cell in the position $h - 1$). Thus we conclude that $\mu([x_n| \ldots |x_{h+2}|x_{h+1}x_h|x_{h-1}| \ldots |x_1]) = [x_n| \ldots |x_1]$ and so $\mu^2([x_n| \ldots |x_1]) = [x_n| \ldots |x_1]$.

Now start with a redundant cell $[x_n| \ldots |x_1]$ of height 0. There is by definition a $y \in M \setminus \{1\}$ with $x_1 = y\Delta_{I(x_1)}$, and $\mu([x_n| \ldots |x_1]) = [x_n| \ldots |x_2|y|\Delta_{I(x_1)}]$. Let w be the normal form for y, then $w\Delta_{I(x_1)}$ is the normal form of x_1. In particular, we see that the image cell is collapsible of height 1 and $\mu([x_n| \ldots |x_2|y|\Delta_{I(x_1)}]) = [x_n| \ldots |x_2|y|\Delta_{I(x_1)}]$. Thus $\mu^2([x_n| \ldots |x_1]) = [x_n| \ldots |x_1]$.

Last, consider a redundant cell $[x_n|\ldots|x_1]$ of height $h > 0$. We have

$$\mu([x_n|\ldots|x_1]) = [x_n|\ldots|x_{h+2}|a|b|x_h|x_{h-1}|\ldots|x_1],$$

where b is the minimal prefix of w_{h+1} such that bw_h is not a normal form. Therefore, the cell $[b|x_h|x_{h-1}|\ldots|x_1]$ is essential. Furthermore, we see that the normal form of ab is the concatenation of normal forms for a and b since we defined $w_{h+1} = ab$ and, by Lemma 5.1.6, both subwords a and b are then normal forms. So the image cell is collapsible of height $h + 1$, and

$$\mu([x_n|\ldots|x_{h+2}|a|b|x_h|x_{h-1}|\ldots|x_1]) = [x_n|\ldots|x_{h+2}|ab|x_h|x_{h-1}|\ldots|x_1].$$

So we have again $\mu^2([x_n|\ldots|x_1]) = [x_n|\ldots|x_1]$. This shows that μ is an involution.

Next, we are going to show that μ is $\mathbb{Z}$-compatible. Consider a redundant cell $\underline{x} := [x_n|\ldots|x_1]$. Clearly, $d_{h+1}([x_n|\ldots|x_{h+2}|a|b|x_h|x_{h-1}|\ldots|x_1]) = \underline{x}$ if $\mathrm{ht}(\underline{x}) = h > 0$ and $d_1([x_n|\ldots|x_2|y|\Delta_{I(x_1)}]) = \underline{x}$ for $\mathrm{ht}(\underline{x}) = 0$, and it is easy to see that none of the other d_i's produces $\underline{x}$ using cancellativity of M (cf. also the proof of Lemma 3.4.7). This shows that μ is a $\mathbb{Z}$-compatible matching.

Now we proceed by showing that this matching is noetherian. Note that μ does not change the product of all entries of the cell. The maps d_i for $1 \leq i \leq n-1$ also preserve the product of the cell $[x_n|\ldots|x_1]$.

Assume now we have an infinite sequence of redundant cells $\underline{x}_1, \underline{x}_2, \ldots$, such that $\underline{x}_{i+1}$ is $d_{k_i}(\mu(\underline{x}_i))$ for some k_i, and we may assume that $k_i \neq h+1$, so that $\underline{x}_{i+1} \neq \underline{x}_i$. Let $\underline{x}_1 = [x_n|\ldots|x_1]$. Then only finitely many k_i can be 0 or n since d_0 and d_n strictly lower the S-length of the product. So we can directly assume there are only $k_i \in \{1, 2, \ldots, n-1\}$. Next, we observe that $\widetilde{I}(\mu(\underline{x}_i)) = \widetilde{I}(\underline{x}_i)$ since $\underline{x}_i$ is redundant. Also, $\widetilde{I}(d_j(\underline{y})) \geq \widetilde{I}(\underline{y})$ for $1 \leq j \leq n-1$ and $\underline{y}$ of dimension n as a consequence of Lemma 5.1.9. Furthermore, it follows from Lemma 5.1.9 that $\widetilde{I}(\underline{x}_i) \leq \widetilde{I}(x_n \ldots x_1)$. Note moreover that, since the product of entries of $\underline{x}_i$ is constantly $x_n \ldots x_1$, the tuple $\widetilde{I}(\underline{x}_i)$ may contain at most $N_S(x_n \ldots x_1)$ non-empty sets. Now there are only finitely many tuples of subsets of S which are smaller than $\widetilde{I}(x_n \ldots x_1)$ in the described order and have bounded length. Therefore, $\widetilde{I}(\underline{x}_i)$ must be constant from a certain step on, and we may directly assume that it is constant for our sequence. In particular, each d_{k_i} multiplies two elements in the way such that the normal form of the product is the concatenation of the normal forms of the factors. We are going to investigate the cases where it can happen.

First, let $\underline{y} = [y_n|\ldots|y_1]$ be a redundant cell of height $h > 0$, and let

$$\mu([y_n|\ldots|y_1]) = [y_n|\ldots|y_{h+2}|a|b|y_h|y_{h-1}|\ldots|y_1]$$

be its collapsible partner. Note that the cell $[a|b|y_h|y_{h-1}|\ldots|y_1]$ is collapsible of height $h+1$ as shown before. Thus, $d_j(\mu(\underline{y}))$ is also collapsible for $h+3 \leq j \leq n$. Furthermore, by definition of the height, $y_j y_{j-1}$ does not have the product of normal forms as a normal form for $2 \leq j \leq h$, and same is true for by_h. So the differentials d_j for $1 \leq j \leq h$ do not satisfy the condition above. So the only possible successor of $\underline{y}$ in our sequence is $[y_n|\ldots|y_{h+2}a|b|y_h|y_{h-1}|\ldots|y_1]$, and its height is at least $h+1 > h$.

Now let $\underline{y} = [y_n | \ldots | y_1]$ be a redundant cell of height 0, set $I = I(y_1)$ and $z \in M \setminus \{1\}$ with $y_1 = z\Delta_I$. So we have $\mu([y_n | \ldots | y_1]) = [y_n | \ldots | y_2 | z | \Delta_I]$ which is collapsible of height 1. Again, $d_j(\mu(\underline{y}))$ is also collapsible for $3 \le j \le n$. So the only possible successor $\underline{y}$ in our sequence is $[y_n | \ldots | y_2 z | \Delta_I]$ of height at least $1 > 0$.

In any case, we see that the height must strictly increase in each step. On the other hand, the height is bounded by n, so that the sequence has to stabilize. So we have shown that the matching is noetherian, and this completes the proof of the proposition. $\qquad \square$

This noetherian matching gives us the following complex computing the homology of Artin monoids.

Corollary 5.1.10. *Let $M(S)$ be an Artin monoid. Then its homology can be computed as the homology of the following chain complex (C_*^θ, d^θ): The module C_n^θ is a free $\mathbb{Z}$-module with basis*

$$[x_n | \ldots | x_1]$$

subject to the conditions (a)-(c) of Definition 5.1.4. If we denote again by $w_i \in \mathcal{D}^*$ the Brieskorn-Saito normal forms of x_i, we can formulate the conditions as follows:

(a) $x_1 \in \mathcal{D}$,

(b) The word $w_{i+1}w_i$ is not the normal form of $x_{i+1}x_i$ for every $1 \le i \le n-1$,

(c) For every $1 \le i \le n-1$, any proper (right) prefix of $w_{i+1}w_i$ is a normal form.

We want to show that this matching can also be lifted to the CW complex BM.

Corollary 5.1.11. *Let $M(S)$ be an Artin monoid. Then the space $BM(S)$ is homotopy equivalent to a CW complex with cells of the form*

$$[x_n | \ldots | x_1]$$

in dimension n, subject to the conditions (a)-(c) of Definition 5.1.4. If we denote again by $w_i \in \mathcal{D}^*$ the Brieskorn-Saito normal forms of x_i, we can formulate the conditions as follows:

(a) $x_1 \in \mathcal{D}$,

(b) The word $w_{i+1}w_i$ is not the normal form of $x_{i+1}x_i$ for every $1 \le i \le n-1$,

(c) For every $1 \le i \le n-1$, any proper (right) prefix of $w_{i+1}w_i$ is a normal form.

Proof. We want to apply Lemma 3.4.7 to obtain an acyclic matching on BM out of μ. This is possible since M is cancellative and has no non-trivial invertible elements (e.g. due to the fact that all relations in an Artin monoid are homogeneous.), and furthermore $\mu(\underline{x}) = d_{\text{ht}(\underline{x})}(\underline{x})$ for any μ-collapsible cell $\underline{x}$, and the height $\text{ht}(\underline{x})$ of a μ_1-collapsible cell satisfies $1 \le \text{ht}_1(\underline{x}) \le n-1$. In order to apply Theorem 3.3.8, we have to check that the

universal grading for this matching is compact. We will use the Criterion 3.3.9. Define the map

$$\begin{aligned} \psi\colon BM^{(*)} &\rightarrow \mathbb{N} \\ [x_n|\ldots|x_1] &\mapsto N_S(x_n\ldots x_1) \end{aligned}$$

First, we observe that this is a map of posets: Taking boundaries either leaves the value of ψ constant (if it is d_i for $1 \leq i \leq n-1$) or decreases the value (for $i \in \{0,n\}$). Moreover, by definition of μ, the value of ψ is the same on the elements matched by μ. Last, there are only finitely many elements of $BM^{(*)}$ such that the norm of the product over all entries does not exceed a given value. Thus, the compactness condition is also satisfied. $\qquad\square$

5.2 Squier Complex for Artin Monoids

In this section, we define a noetherian matching μ_1 on the bar complex of an Artin monoid and a further noetherian matching μ_2 on the obtained chain complex, so that the resulting chain complex is related to the one defined by Squier ([56]). We start by defining μ_1 and proving that it is a noetherian matching.

Let $M := M(S)$ be again an Artin monoid with Artin-Coxeter generating set S. Let $W(S)$ be the corresponding Coxeter group. Set

$$S^f = \{I \subset S | W(I) \text{ is finite}\}.$$

Furthermore, set $\mathcal{D} = \{\Delta_I := \text{l-lcm}(I) | \varnothing \neq I \in S^f\}$.

Definition 5.2.1. *We call an n-cell $[x_n|\ldots|x_1]$ of the bar complex μ_1-**essential** if for any $1 \leq k \leq n$, the product $x_k\ldots x_1$ lies in $\mathcal{D}$. Define $\mu_1([x_n|\ldots|x_1]) = [x_n|\ldots|x_1]$ for every essential cell $[x_n|\ldots|x_1]$.*

*For an arbitrary cell $[x_n|\ldots|x_1]$, we define its μ_1-**height** by*

$$\text{ht}_1([x_n|\ldots|x_1]) = \max\{j | [x_j|\ldots|x_1] \text{ is essential}\}$$

If $x_1 \notin \mathcal{D}$, set $\text{ht}_1([x_n|\ldots|x_1]) = 0$. (Set for further use $x_0 = 1$.)

For an n-cell $[x_n|\ldots|x_1]$ of height h and $1 \leq k \leq h$, define $I_k \subset S$ to be the unique subset such that $x_k\ldots x_1 = \Delta_{I_k}$. Note that $I_1 \subsetneq I_2 \subsetneq \ldots \subsetneq I_h$, and, furthermore, $I_h \subset I(x_{h+1}x_h\ldots x_1) = I(x_{h+1}\Delta_{I_h})$.

*Define an n-cell $[x_n|\ldots|x_1]$ of height $h < n$ to be μ_1-**collapsible** if $I(x_{h+1}x_h\ldots x_1) = I_h$ holds. In this case, set*

$$\mu_1([x_n|\ldots|x_1]) = [x_n|\ldots|x_{h+2}|x_{h+1}x_h|x_{h-1}|\ldots|x_1].$$

*Define an n-cell $[x_n|\ldots|x_1]$ of height $h < n$ to be μ_1-**redundant** if $I_h \subsetneq J := I(x_{h+1}x_h\ldots x_1)$. In this case, there exists a unique $y \in M \setminus \{1\}$ such that $x_{h+1}\Delta_{I_h} = y\Delta_J$; furthermore, there is a unique $z \in M \setminus \{1\}$ such that $\Delta_J = z\Delta_{I_h}$. Define*

$$\mu_1([x_n|\ldots|x_1]) = [x_n|\ldots|x_{h+2}|y|z|x_h|x_{h-1}|\ldots|x_1].$$

In particular, if $[x_n|\ldots|x_1]$ is of height 0, i.e., if $x_1 \notin \mathcal{D}$, we want to define this cell to be redundant (according to our convention, $I(x_0) = \varnothing \subsetneq I(x_1)$). There is then a unique $y \in M \setminus \{1\}$ such that $x_1 = y\Delta_{I(x_1)}$; in this case, $z = \Delta_{I(x_1)}$. We define

$$\mu_1([x_n|\ldots|x_1]) = [x_n|\ldots|x_2|y|\Delta_{I(x_1)}].$$

As promised, we are now going to prove the following proposition.

Proposition 5.2.2. *For any Artin monoid M, the map $\mu_1\colon \Omega_* \to \Omega_*$ defined as above gives a noetherian, $\mathbb{Z}$-compatible matching on the bar complex.*

Proof. First, we are going to show that μ_1 is an involution. We begin with a collapsible n-cell $\underline{x} = [x_n|\ldots|x_1]$ of height h. Then

$$\mu_1([x_n|\ldots|x_1]) = [x_n|\ldots|x_{h+2}|x_{h+1}x_h|x_{h-1}|\ldots|x_1]$$

is of height $h-1$, since $x_k \ldots x_1 = \Delta_{I_k}$ for $1 \le k \le h-1$ and $(x_{h+1}x_h)(x_{h-1}\ldots x_1) \notin \mathcal{D}$ by definition. Now since $\underline{x}$ was collapsible, we know that $I((x_{h+1}x_h)(x_{h-1}\ldots x_1)) = I_h \supsetneq I_{h-1}$. Hence, $\mu_1(\underline{x})$ is a redundant cell of height $h-1$. Furthermore, we have $\Delta_{I_h} = x_h\Delta_{I_{h-1}}$ and $x_{h+1}x_h\Delta_{I_{h-1}} = x_{h+1}\Delta_{I_h}$.

This implies $\mu_1^2(\underline{x}) = [x_n|\ldots|x_{h+2}|x_{h+1}|x_h|x_{h-1}|\ldots|x_1]$.

Next, consider a redundant n-cell $\underline{x} = [x_n|\ldots|x_1]$ of height $h > 0$ with

$$\mu_1([x_n|\ldots|x_1]) = [x_n|\ldots|x_{h+2}|y|z|x_h|x_{h-1}|\ldots|x_1].$$

Then we know that $x_k \ldots x_1 = \Delta_{I_k}$ for $1 \le k \le h$. Furthermore, by definition, $zx_h \ldots x_1 = z\Delta_{I_h} = \Delta_J$ and $yzx_h \ldots x_1 = x_{h+1}x_h \ldots x_1 \notin \mathcal{D}$. Thus, the cell $\mu_1(\underline{x})$ has height $h+1$. Moreover, we have $I(yzx_h \ldots x_1) = I(x_{h+1}x_h \ldots x_1) = J$, so that $\mu_1(\underline{x})$ is collapsible.

Last, we consider the case of a redundant n-cell $\underline{x} = [x_n|\ldots|x_1]$ of height $h = 0$. We know that $\mu_1([x_n|\ldots|x_1]) = [x_n|\ldots|x_2|y|\Delta_{I(x_1)}]$. The height of this new cell is at least 1 since $\Delta_{I(x_1)} \in \mathcal{D}$. Moreover, it is of height exactly 1 since

$$I(y\Delta_{I(x_1)}) = I(x_1) = I(\Delta_{I(x_1)}),$$

and since by assumption $y\Delta_{I(x_1)} = x_1 \notin \mathcal{D}$. This implies that the cell $\mu_1([x_n|\ldots|x_1])$ is collapsible of height 1, and it is mapped by μ_1 to

$$\mu_1^2([x_n|\ldots|x_1]) = \mu_1([x_n|\ldots|x_2|y|\Delta_{I(x_1)}]) = [x_n|\ldots|x_1].$$

We conclude that $\mu_1^2(\underline{x}) = [x_n|\ldots|x_{h+2}|x_{h+1}|x_h|x_{h-1}|\ldots|x_1]$. This shows that μ_1 is indeed an involution.

Now we observe that μ_1 is $\mathbb{Z}$-compatible. Consider a redundant cell $\underline{x} := [x_n|\ldots|x_1]$. Clearly, $d_{h+1}([x_n|\ldots|x_{h+2}|y|z|x_h|x_{h-1}|\ldots|x_1]) = \underline{x}$ if $\mathrm{ht}_1(\underline{x}) = h$ and it is easy to see that none of the other d_i's produces $\underline{x}$ due to the cancellativity of M (cf. proof of the Lemma 3.4.7). This shows that μ_1 is a $\mathbb{Z}$-compatible matching.

Finally, we are going to show that the matching μ_1 is noetherian. Suppose we have an infinite chain of redundant cells $\underline{x}_1, \underline{x}_2, \ldots$ such that $\underline{x}_{i+1}$ is $d_{k_i}(\mu(\underline{x}_i))$ for some k_i, and we may assume that $k_i \neq h+1$ so that $\underline{x}_{i+1} \neq \underline{x}_i$. Then only finitely many k_i can be 0 or n since d_0 and d_n strictly lower the S-length of the product. So we can directly assume there are only $k_i \in \{1, 2, \ldots, n-1\}$. We look for possible successors of a redundant $\underline{x} = [x_n | \ldots | x_1]$ of height h. For $h+3 \leq k \leq n$, the cell $d_k([x_n | \ldots | x_{h+2} | y | z | x_h | x_{h-1} | \ldots | x_1])$ is obviously collapsible, as well as for $1 \leq k \leq h$. The cell $d_{h+2}([x_n | \ldots | x_{h+2} | y | z | x_h | x_{h-1} | \ldots | x_1])$ may or may not be redundant, so it is the only possible successor. In any case, note that $\text{ht}_1([x_n | \ldots | x_{h+2} y | z | x_h | x_{h-1} | \ldots | x_1]) \geq h+1$. Thus, the height in such a chain must strictly increase, so the sequence of redundant cells as above must stabilize after finitely many steps. $\qquad\square$

Remark 5.2.3. *By Theorem 3.2.4, we know that the complex $(C_*^{\theta_1}, d_*^{\theta_1})$ computes the homology of an Artin monoid M, where $C_n^{\theta_1}$ has as a $\mathbb{Z}$-basis the μ_1-essential n-cells, and $d_*^{\theta_1} = \theta_1^\infty \circ d$. Now if $\underline{x} = [x_n | \ldots | x_1]$ is an essential cell, it is clear that $d_i(\underline{x})$ is essential for $1 \leq i \leq n$, while $d_0(\underline{x})$ may or may not be essential. Thus, we have*

$$d_*^{\theta_1}([x_n | \ldots | x_1]) = d(\underline{x}) - [x_n | \ldots | x_2] + \theta_1^\infty([x_n | \ldots | x_2])$$

Note that the summands of $d_*^{\theta_1}(\underline{x})$ are either $\pm d_i(\underline{x})$ for $1 \leq i \leq n-1$ or the product of their entries have smaller S-length than such of $\underline{x}$. We will need this description later.

Furthermore, note that any μ_1-essential cell $[x_n | \ldots | x_1]$ is uniquely characterized by the sequence $I_n \supsetneq I_{n-1} \supsetneq \ldots \supsetneq I_1$. We denote the set of such cells in dimension n by $\Omega_n^{\theta_1}$. (This is a basis for $C_n^{\theta_1}$.)

Now we are going to define a noetherian matching μ_2 on the obtained chain complex $(C_*^{\theta_1}, d_*^{\theta_1})$ making it smaller again. For this, choose any linear order $<$ on the set S. We now describe the essential, collapsible and redundant cells of the matching μ_2.

Definition 5.2.4. *Let $[x_n | \ldots | x_1]$ be an n-cell in $\Omega_n^{\theta_1}$. We say $[x_n | \ldots | x_1]$ to be μ_2-**essential** if for any $1 \leq k \leq n$, $I_k \setminus I_{k-1} = \{a_k\}$ and $a_k = \max I_k$. (Here, we set $I_0 = \varnothing$.) Define $\mu_2([x_n | \ldots | x_1]) = [x_n | \ldots | x_1]$ for an essential cell $[x_n | \ldots | x_1]$.*

*For an arbitrary cell $[x_n | \ldots | x_1]$, we define its μ_2-**height** by*

$$\text{ht}_2([x_n | \ldots | x_1]) = \max\{j \mid [x_j | \ldots | x_1] \text{ is essential}\}$$

If $\#I_1 > 1$, set $\text{ht}_2([x_n | \ldots | x_1]) = 0$.

*Define an n-cell $[x_n | \ldots | x_1]$ of height $h < n$ to be μ_2-**collapsible** if $\max I_{h+1} = \max I_h$ holds. In this case, set*

$$\mu_2([x_n | \ldots | x_1]) = [x_n | \ldots | x_{h+2} | x_{h+1} x_h | x_{h-1} | \ldots | x_1].$$

The characterizing sequence of the new element is

$$I_n \supsetneq \ldots \supsetneq I_{h+2} \supsetneq I_{h+1} \supsetneq I_{h-1} \supsetneq \ldots \supsetneq I_1.$$

*Define an n-cell $[x_n | \ldots | x_1]$ of height $h < n$ to be μ_2-**redundant** if $b := \max I_{h+1} > a_h = \max I_h$. Observe that in this case $\#I_{h+1} \geq 2 + \#I_h$ since otherwise the cell would*

have at least height $h + 1$. Thus there exist $u, v \in M \setminus \{1\}$ such that $\Delta_{I_{h+1}} = u\Delta_{I_h \cup \{b\}}$ and $\Delta_{I_h \cup \{b\}} = v\Delta_{I_h}$. Define

$$\mu_2([x_n | \dots | x_1]) = [x_n | \dots | x_{h+2} | u | v | x_h | x_{h-1} | \dots | x_1].$$

Note that the characterizing sequence of the new element is

$$I_n \supsetneq \dots \supsetneq I_{h+2} \supsetneq I_{h+1} \supsetneq I_h \cup \{b\} \supsetneq I_h \supsetneq I_{h-1} \supsetneq \dots \supsetneq I_1.$$

Observe furthermore that $\Delta_{I_{h+1}} = u\Delta_{I_h \cup \{b\}} = uv\Delta_{I_h}$ implies $x_{h+1} = uv$.

We are going to prove that μ_2 is a noetherian matching on $(C_*^{\theta_1}, d_*^{\theta_1})$. This works quite the same way as the last proof.

Proposition 5.2.5. *For any Artin monoid M, the map $\mu_2 \colon \Omega_*^{\theta_1} \to \Omega_*^{\theta_1}$ defined as above gives a noetherian, $\mathbb{Z}$-compatible matching on $(C_*^{\theta_1}, d_*^{\theta_1})$.*

Proof. First, we are going to show that μ_2 is an involution. We begin with a collapsible n-cell $\underline{x} = [x_n | \dots | x_1]$ of height h. Then

$$\mu_2([x_n | \dots | x_1]) = [x_n | \dots | x_{h+2} | x_{h+1} x_h | x_{h-1} | \dots | x_1]$$

is of height $h - 1$, since $I_k \setminus I_{k-1} = \{a_k\}$ and $a_k = \max I_k$ for $1 \le k \le h - 1$ and $\#(I_{h+1} \setminus I_{h-1}) \ge 2$. Since $\underline{x}$ was collapsible of height h, we know that $\max I_{h+1} = a_h > \max I_{h-1} = a_{h-1}$, so that $\mu_2(\underline{x})$ is a redundant cell of height $h - 1$. Furthermore, we have $I_h = I_{h-1} \cup \{a_h\}$ and $\Delta_{I_h} = x_h \Delta_{I_{h-1}}$ as well as $\Delta_{I_{h+1}} = x_{h+1} \Delta_{I_h}$. This implies by definition $\mu_2([x_n | \dots | x_1]) = [x_n | \dots | x_{h+2} | x_{h+1} | x_h | x_{h-1} | \dots | x_1]$.

Next, consider a redundant n-cell $\underline{x} = [x_n | \dots | x_1]$ of height h with $\mu_2([x_n | \dots | x_1]) = [x_n | \dots | x_{h+2} | u | v | x_h | x_{h-1} | \dots | x_1]$ as defined above. Again, we have $I_k \setminus I_{k-1} = \{a_k\}$ and $a_k = \max I_k$ for $1 \le k \le h$. Furthermore, we know that $v x_h \dots x_1 = \Delta_{I_h \cup \{b\}}$ and $b = \max(I_h \cup \{b\}) > \max I_h$ by definition. In addition, we have that $b = \max(I_h \cup \{b\}) = \max I_{h+1}$, so that $\mu_2(\underline{x})$ is a collapsible cell of height $h + 1$.

We conclude that $\mu_2^2(\underline{x}) = [x_n | \dots | x_{h+2} | x_{h+1} | x_h | x_{h-1} | \dots | x_1]$. This shows that μ_2 is indeed an involution.

Now we observe that μ_2 is $\mathbb{Z}$-compatible. Consider a redundant cell $\underline{x} := [x_n | \dots | x_1]$. Clearly, $d_{h+1}([x_n | \dots | x_{h+2} | u | v | x_h | x_{h-1} | \dots | x_1]) = \underline{x}$ if $\mathrm{ht}_2(\underline{x}) = h$ and it is easy to see that none of the other d_i-summands for $1 \le i \le n$ produces $\underline{x}$. Since the other summands of $d_*^{\theta_1}$ have smaller S-norm, they cannot coincide with $\underline{x}$. This shows that μ_2 is a $\mathbb{Z}$-compatible matching.

Finally, we are going to show that the matching μ_2 is noetherian. Suppose we have an infinite sequence of redundant n-cells $\underline{x}_1, \underline{x}_2, \dots$, such that $\underline{x}_{i+1}$ is a summand of $d_*^{\theta_1}(\mu_2(\underline{x}_i))$. We may assume that $\underline{x}_{i+1} \ne \underline{x}_i$. Moreover, we may assume that the S-length of the product of all entries is constant, since it is non-increasing and finite. Thus, we may assume that $\underline{x}_{i+1} = d_{k_i}(\mu_2(\underline{x}_i))$ with $k_i \in \{1, 2, \dots, n\}$.

Define a (lexicographic) order on characterizing sequences as follows: Let

$$(A_m \supsetneq A_{m-1} \supsetneq \dots \supsetneq A_1) < (B_l \supsetneq B_{l-1} \supsetneq \dots \supsetneq B_1)$$

hold if there exist $s < \max\{m,l\}$ such that $\max A_i = \max B_i$ for all $1 \le i \le s-1$ and $\max A_s < \max B_s$, or if $\max A_i = \max B_i$ for all $1 \le i \le \max\{m,l\}$ and there is a $t < \max\{m,l\}$ such that $\#A_i = \#B_i$ for $1 \le i \le t-1$ and $\#A_t > \#B_t$. (For $i > m$, set $A_i = A_m$, similar for B_i. The opposite directions of the inequality signs may be a bit confusing at the beginning, but this is the definition we will need.)

We consider the characterizing sequences for the successors of a redundant cell $\underline{x} = [x_n|\ldots|x_1]$ of height h. For $h+3 \le k \le n$, the cell $d_k([x_n|\ldots|x_{h+2}|u|v|x_h|x_{h-1}|\ldots|x_1])$ is obviously collapsible and thus not a successor of $\underline{x}$. For $1 \le k \le h$, the characterizing sequence of $d_k([x_n|\ldots|x_{h+2}|u|v|x_h|x_{h-1}|\ldots|x_1])$ is larger than the one of $\underline{x}$ since both coincide for $1 \le i \le k-1$ and for $i = k$ we have $\max I_{k+1} = a_{k+1} > \max I_k = a_k$. (This holds also for $k = h$ since also $b = \max(I_h \cup \{b\}) > \max I_h$.) The cell $d_{h+2}([x_n|\ldots|x_{h+2}|u|v|x_h|x_{h-1}|\ldots|x_1]) = [x_n|\ldots|x_{h+2}u|v|x_h|x_{h-1}|\ldots|x_1]$ has the characterizing sequence

$$I_n \supsetneq \ldots \supsetneq I_{h+2} \supsetneq I_h \cup \{b\} \supsetneq I_h \supsetneq I_{h-1} \supsetneq \ldots \supsetneq I_1$$

This sequence has in each place the same maximum as the original one and $\#(I_h \cup \{b\}) < \#I_{h+1}$, so that this sequence is again larger than the one of $\underline{x}$.

Thus, the characterizing sequence in such a chain must strictly increase, so the sequence of redundant cells as above must stabilize after finitely many steps. $\qquad\square$

Note that the essential cells in dimension k of the new complex are in one-to-one correspondence with the k-element subsets of S which lie in S^f. This property is shared by the following complex, found by Squier:

Proposition 5.2.6. *([56], Theorem 7.5) Let $M(S)$ be an Artin monoid, and let S be linearly ordered. Then the following complex computes the homology of $M(S)$: The free modules C_k are generated by the k-element subsets of S which lie in S^f. The value of the differential on the generator $[I] = [a_1 < a_2 < \ldots < a_k]$ is given by*

$$\partial([I]) = \sum_{i=1}^{k} (-1)^{i-1} \cdot \left(\sum_{uv = \Delta_I \Delta_{I\setminus\{a_i\}}^{-1}} (-1)^{N_S(u)} \right) [I \setminus \{a_i\}].$$

We will call this complex the Squier complex.

Remark 5.2.7. *There is at least one further complex in the literature computing the homology of Artin monoids which has the same number of generators as the Squier complex: It comes from a space homotopy equivalent to BM and is often called* Salvetti complex. *It can be found e.g. in [23], [54], cf. also [17]. It seems to be not written down whether the Salvetti complex and the Squier complex coincide (cf. Introduction of [34]). It is unclear to the author how the differentials in the complex obtained after applying the matching μ_2 of this section compare to those of the Salvetti complex and to those of the Squier complex.*

These complexes seem to be the smallest written down in this generality in the literature. Note that the generalized Charney-Meier-Whittlesey complex of Section 4.8 is larger, but has much simpler differentials.

5.3 A Reformulation of the $K(\pi, 1)$-conjecture

The aim of this section is to reprove a theorem by N. Dobrinskaya ([33]) claiming that the $K(\pi, 1)$-conjecture for an Artin group $G(S)$ is equivalent to the question whether the inclusion $BM(S) \to BG(S)$ is a homotopy equivalence. Her proof is rather lengthy and uses a complicated machinery of configuration spaces. It seems that our proof is, though in a sense less geometrical, yet more transparent and less involved.

Let $M := M(S)$ be an Artin monoid with Artin-Coxeter generating set S and let $G(S)$ be the corresponding group. Let $W(S)$ be the corresponding Coxeter group (cf. Definition 4.1.7). For $I \subset S$, let $W(I)$ be the Coxeter group given by the restriction of the Coxeter matrix to I. (It is well-known that $W(I)$ is the subgroup of $W(S)$ generated by I). Set furthermore again

$$S^f = \{I \subset S | W(I) \text{ is finite}\}$$

Theorem 5.3.1 ([18]). *We have a natural isomorphism $G(S) \cong \mathrm{colim}_{T \in S^f} G(T)$.*

We want to observe that the same holds for monoids.

Lemma 5.3.2. *The canonical map $\mathrm{colim}_{T \in S^f} M(T) \to M(S)$ is an isomorphism.*

Proof. Let N be any monoid, and assume we have compatible monoid homomorphisms $\varphi_T \colon M(T) \to N$ for $T \in S^f$. Since for each $a \in S$, we have $\{a\} \in S^f$, we can set $\varphi(a) := \varphi_{\{a\}}(a)$. We now want to show that φ defines a monoid homomorphism $M(S) \to N$. Recall that all relations in $M(S)$ are of the type

$$\langle a, b \rangle^{m_{a,b}} = \langle b, a \rangle^{m_{a,b}}$$

whenever $m_{a,b}$ is finite. But if $m_{a,b}$ is finite, the corresponding dihedral group associated to $M(\{a, b\})$ is finite, thus $\{a, b\} \in S^f$. Since $\varphi_{\{a,b\}}(a) = \varphi_{\{a\}}(a) = \varphi(a)$ and similar for b, we know that the elements $\varphi(a), \varphi(b) \in N$ satisfy

$$\langle \varphi(a), \varphi(b) \rangle^{m_{a,b}} = \langle \varphi(b), \varphi(a) \rangle^{m_{a,b}}$$

since $\varphi_{\{a,b\}}$ is a monoid homomorphism. So φ is a well-defined monoid homomorphism, and it is compatible with each φ_T since they coincide on T. Moreover, since the values of φ on S are fixed by the family φ_T, the monoid homomorphism φ is unique. So $M(S)$ has the universal property of the colimit, and this implies the claim. $\square$

We will need the Grothendieck construction for a functor $F \colon \mathcal{C} \to \mathrm{Cat}$, where $\mathcal{C}$ is a small category and Cat is the category of small categories, like described in [58]. We follow Thomason for the exposition. To such a functor, we assign a category $\mathcal{C} \int F$. Its objects are pairs (C, x), where C is an object in $\mathcal{C}$ and x is an object in $F(C)$. A morphism from (C_1, x_1) to (C_2, x_2) is given by a map $c \colon C_1 \to C_2$ in $\mathcal{C}$ and a map $\varphi \colon F(c)(x_1) \to x_2$ in the category $F(C_2)$. The composition with a further morphism $(c', \varphi') \colon (C_0, x_0) \to (C_1, x_1)$ is given by

$$(c, \varphi) \circ (c', \varphi') = (cc', \varphi \circ F(c)(\varphi')).$$

Note that the construction is functorial: a natural transformation $\alpha\colon F \Rightarrow F'$ induces a functor $\mathcal{C} \int \alpha\colon \mathcal{C} \int F \to \mathcal{C} \int F'$, given by $(\mathcal{C} \int \alpha)\,(C, x) = (C, \alpha(C)(x))$ on objects and by $(\mathcal{C} \int \alpha)\,(c, \varphi) = (c, \alpha(C_2)(\varphi))$ on morphisms. One checks that this defines a functor from the functor category $\mathrm{Fun}(\mathcal{C}, \mathrm{Cat})$ into Cat.

For the exact definition of a homotopy colimit, see e.g. [8]. We will need, besides the theorem below, mainly the following homotopy lemma:

Lemma 5.3.3 ([8], XII, 3.7 and 4.2)**.** *Let X, Y be two functors from a small category $\mathcal{C}$ to simplicial sets, and let $\psi\colon X \to Y$ be a natural transformation. Then there exists an induced map $\mathrm{hocolim}\,\psi\colon \mathrm{hocolim}\, X \to \mathrm{hocolim}\, Y$ making $\mathrm{hocolim}$ into a functor. Furthermore, if for all objects C in $\mathcal{C}$, the map $\psi(C)\colon X(C) \to Y(C)$ is a weak homotopy equivalence, then the induced map $\mathrm{hocolim}\,\psi$ is also a weak homotopy equivalence.*

We will use the following homotopy colimit theorem by Thomason:

Theorem 5.3.4 ([58])**.** *Let $F\colon \mathcal{C} \to \mathrm{Cat}$ be a functor. Then there is a natural weak homotopy equivalence*

$$\eta(F)\colon \mathrm{hocolim}\, NF \to N(\mathcal{C} \int F)$$

of simplicial sets.

Combining these two results, we obtain:

Proposition 5.3.5. *Let $F, G\colon \mathcal{C} \to \mathrm{Cat}$ be two functors starting from a small category $\mathcal{C}$, and let $\psi\colon F \Rightarrow G$ be a natural transformation between them such that*

$$N(\psi(C))\colon NF(C) \to NG(C)$$

is a weak homotopy equivalence for each object C of $\mathcal{C}$. Then the induced map of simplicial sets $N(\mathcal{C} \int \psi)\colon N(\mathcal{C} \int F) \to N(\mathcal{C} \int G)$ is a weak homotopy equivalence.

We will now apply this proposition to our situation. We consider the functors

$$M(-), G(-)\colon S^f \to \mathrm{Cat}$$

associating to $T \in S^f$ the corresponding Artin monoids and Artin groups, respectively. Here, the category S^f means the category associated to the poset S^f with usual inclusion as ordering. There is a natural transformation $i\colon M(-) \to G(-)$ given by the canonical map. By [10], the Artin monoids of finite type satisfy the Ore condition and are cancellative. By [14], Ch. X, §4, we know that for a cancellative monoid M satisfying the Ore condition and its associated group G, the Tor-term $\mathrm{Tor}_n^{\mathbb{Z}[M]}(\mathbb{Z}[G], \mathbb{Z})$ vanishes for all $n > 0$. Now we will use the following proposition of Fiedorowicz:

Proposition 5.3.6 ([35])**.** *Let M be a monoid and let G be its associated group. Then, the following are equivalent:*

1. $\pi_k(BM) = 0$ for all $k \geq 2$.

2. The map $BM \to BG$ is a homotopy equivalence.

3. $\mathrm{Tor}_n^{\mathbb{Z}[M]}(\mathbb{Z}, \mathbb{Z}[G]) = 0$ for all $n > 0$

So we know that the inclusion $BM(T) \to BG(T)$ is a homotopy equivalence for $T \in S^f$. Altogether, we have proven:

Corollary 5.3.7. *The map $N(S^f \int M(-)) \to N(S^f \int G(-))$ induced by inclusion is a weak homotopy equivalence.*

Now we are going to describe $S^f \int M(-)$ and $S^f \int G(-)$ more concretely. Since each $M(T)$ and $G(T)$ has exactly one object, the set of objects of either Grothendieck construction is exactly S^f. There can be a map from T to T' only if $T \subset T'$. Each such map is given by a self-map of the only object of $M(T')$, so we have

$$S^f \int M(-)(T, T') = \begin{cases} M(T'), & \text{if } T \subset T' \\ \varnothing, & \text{else.} \end{cases}$$

Note that the composition is given by the monoid multiplication. The category $S^f \int G(-)$ has a completely analogous description. We are now going to show:

Proposition 5.3.8. *The space $BM(S)$ is homotopy equivalent to $|N(S^f \int G(-))|$.*

Before proving the Proposition, we are going to point out why this shows the desired equivalence. It follows from [18], Corollary 3.2.4, that the $K(\pi, 1)$-conjecture for $G(S)$ is equivalent to the following:

Conjecture 5.3.9 ([18]). *For any Artin group $G(S)$, the space $|N(S^f \int G(-))|$ is homotopy equivalent to $BG(S)$.*

We can now use the results proven above to conclude the following theorem, first proven (by completely different means) by N. Dobrinskaya.

Theorem 5.3.10. *(cf. [33]) The inclusion $BM(S) \to BG(S)$ is a homotopy equivalence if and only if the space $|N(S^f \int G(-))|$ is homotopy equivalent to $BG(S)$.*

Proof. If the inclusion $BM(S) \to BG(S)$ is a homotopy equivalence, Conjecture 5.3.9 holds for $G(S)$ by Proposition 5.3.8.

For the other implication, we use again Proposition 5.3.6: If the Conjecture 5.3.9 holds for $G(S)$, again Proposition 5.3.8 implies that all higher homotopy groups of BM vanish, thus the claim. $\qquad\square$

We still need to show Proposition 5.3.8. As a first step, we are going to show that the matching μ_1 of the last section is also defined on the CW complex $BM(S)$. For short, we write $M := M(S)$.

We want to apply Lemma 3.4.7 to obtain an acyclic matching on BM out of μ_1. This is possible since M is cancellative and has no non-trivial invertible elements (e.g. due to the fact that all relations in an Artin monoid are homogeneous), and furthermore

$\mu_1(\underline{x}) = d_{\mathrm{ht}_1(\underline{x})}(\underline{x})$ for any μ_1-collapsible cell $\underline{x}$, and the height $\mathrm{ht}_1(\underline{x})$ of a μ_1-collapsible cell satisfies $1 \leq \mathrm{ht}_1(\underline{x}) \leq n - 1$.

In order to apply Theorem 3.3.8, we have to check that the universal grading for this matching is compact. We will use the Criterion 3.3.9. Define the map

$$\psi \colon BM^{(*)} \;\rightarrow\; \mathbb{N}$$
$$[x_n|\dots|x_1] \;\mapsto\; N_S(x_n \dots x_1)$$

First, we observe that this is a map of posets: Taking boundaries either leaves the value of ψ constant (if it is d_i for $1 \leq i \leq n - 1$) or decreases the value (for $i \in \{0, n\}$). Moreover, by definition of μ_1, the value of ψ is the same on the elements matched by μ_1. Last, there are only finitely many elements of $BM^{(*)}$ such that the norm of the product over all entries does not exceed a given value. Thus, the compactness condition is also satisfied.

Now we will show that this matching restricts to a certain subcomplex of BM. Observe that for any $I \subset S$, we have $M(I) \subset M$ and $NM(I)$ is a simplicial subset of NM. So we can consider

$$K = \bigcup_{I \in S^f} BM(I) \subset BM(S),$$

realizing the simplicial subset $\bigcup_{I \in S^f} NM(I)$ of NM. First, we observe that all essential cells lie in $K^{(*)}$, a subposet of $BM^{(*)}$. Now if a μ_1-collapsible cell $[x_n|\dots|x_1]$ lies in some $BM(I)^{(*)}$, so does $d_h([x_n|\dots|x_1])$, its redundant partner. On the other hand, if $[x_n|\dots|x_1]$ is μ_1-redundant and lies in $BM(I)^{(*)}$, it is a consequence of the fact that the relations do not change the set of letters of a word that $x_{h+1} = yz$ and $x_{h+1} \in M(I)$ implies $y, z \in M(I)$. So the matching restricts to the subcomplex K, and it automatically satisfies the conditions of the Definition 3.3.4 as well as the compactness condition.

Next, we show that the associated Morse complexes of the matching given by μ_1 and of its restriction to K are the same, and the projections defined in Definition 3.3.7 coincide on K. Observe that the cells of both Morse complexes $(BM)_{\mu_1}$ and K_{μ_1} are in one-to-one correspondence with essential cells of either complex, which coincide. Furthermore, it follows inductively from Definition 3.3.7 that the projections to the Morse complex coincide on K, and this in turn implies that the attaching maps for the Morse complexes coincide. (Here, we also exploit the fact that K is a subcomplex.) Thus we obtain

Proposition 5.3.11. *The inclusion*

$$K = \bigcup_{I \in S^f} BM(I) \hookrightarrow BM(S)$$

is a homotopy equivalence.

Next, we want to show that $\bigcup_{I \in S^f} NM(I) \simeq \mathrm{hocolim}_{S^f} NM(-)$. We prove first the following auxiliary lemma.

Lemma 5.3.12. *The simplicial set $\widetilde{K} := \bigcup_{I \in S^f} NM(I)$ is the colimit of the functor $S^f \to sSet$, given by $J \mapsto NM(J)$.*

Proof. Since there are compatible maps $NM(J) \to \widetilde{K}$, given just by inclusion, we obtain a map of simplicial sets

$$\mathrm{colim}_{J \in S^f} NM(J) \to \widetilde{K} = \bigcup_{I \in S^f} NM(I).$$

This map is obviously surjective. We will now show that it is also injective. Assume there are some $[x], [y] \in \mathrm{colim}_{J \in S^f} NM(J)$ coming from simplices $x \in NM(J_1)_k$, $y \in NM(J_2)_k$ and mapped to the same element in $\bigcup_{I \in S^f} NM(I)$. This implies that there is a simplex $z \in NM(J_1)_k \cap NM(J_2)_k = NM(J_1 \cap J_2)_k$ mapping both to x and to y under corresponding inclusions. This implies exactly that $[x] = [z] = [y] \in \mathrm{colim}_{J \in S^f} NM(J)$, proving the injectivity. $\square$

Now we are ready to show the following proposition.

Proposition 5.3.13. *The functor $S^f \to sSet$ is cofibrant as an object of $\mathrm{Fun}(S^f, sSet)$, where the model structure on the latter is given by levelwise weak equivalences and levelwise fibrations. This implies in particular that $\widetilde{K} := \bigcup_{I \in S^f} NM(I) \cong \mathrm{colim}_{J \in S^f} NM(J)$ has the weak homotopy type of the homotopy colimit of the functor $S^f \to sSet$ given by $I \mapsto NM(I)$. (cf. [43], Proposition 18.9.4)*

Proof. First, we recall that in order to obtain a model structure on $\mathrm{Fun}(S^f, sSet)$ where we have a nice description of cofibrations and which satisfies the conditions above, we need for example S^f to be a direct category. This is fulfilled since the assignment $I \mapsto \#I$ gives a linear extension to an ordinal given by, for example, $\#S$. Thus, Theorem 5.1.3 of [44] assures the existence of such a model structure, and it furthermore gives a characterization of cofibrant objects in this model structure. So we only need to check that for each object $I \in S^f$, the induced map $L_I(NM(-)) \to NM(I)$ is a cofibration, where $L_I(F)$ denotes the I-latching object of functor F. Recall (e.g. from [44]) that $L_I(F)$ is the colimit of the "restriction" of F to the category of all non-identity morphisms with target I. Note that here, this category is exactly the poset of all proper subsets of I. So $L_I(NM(-))$ is by the same argument as in the proof of Lemma 5.3.12 given by

$$\bigcup_{J \subsetneq I} NM(J),$$

and the map is just the inclusion. This yields the claim. $\square$

Proof. (of Proposition 5.3.8) We put together all the steps done so far. In Proposition 5.3.11, we have shown that $K \simeq BM(S)$. Going through the definition, we observe that K is the geometric realization of the simplicial set $\widetilde{K}$. By Proposition 5.3.13, we obtain a weak homotopy equivalence in simplicial sets between $\widetilde{K}$ and $\mathrm{hocolim}_{S^f} NM(-)$. By Theorem 5.3.4, this last simplicial set is weakly homotopy equivalent to $N(S^f \int M(-))$, and by Corollary 5.3.7, this simplicial set is in turn weakly homotopy equivalent to $N(S^f \int G(-))$. After geometric realization, we obtain a true homotopy equivalence $K \simeq |N(S^f \int G(-))|$. This completes the proof. $\square$

Remark 5.3.14. *There are already several applications of discrete Morse theory to hyperplane arrangements in the literature, e.g. in [55], [51], [31]. Recall that the original formulation of $K(\pi, 1)$-conjecture claims that a certain hyperplane arrangement is a $K(G(S), 1)$ for an Artin group $G(S)$.*

6 Coxeter Groups and the Orthogonal Group

In this chapter, we are going to investigate the existence of factorability structures on orthogonal groups and on some Coxeter groups.

6.1 Basics on Orthogonal Group

We will use the insights of Brady and Watt ([9]) to define a factorability structure on $O(V)$. First, we recollect their results and methods.

Let V be an n-dimensional vector space over a field $\mathbb{F}$, $\operatorname{char}(\mathbb{F}) \neq 2$, with an anisotropic symmetric bilinear form $\langle -, - \rangle$. Recall that a symmetric bilinear form is called anisotropic if $\langle x, x \rangle = 0$ implies $x = 0$. We consider the group $O(V)$ of orthogonal transformations of V.

Definition 6.1.1. *[9] For $A \in O(V)$, define the **fixed space** of A by $F(A) = \operatorname{Ker}(A - \operatorname{id})$ and the **moved space** of A by $M(A) = \operatorname{im}(A - \operatorname{id})$.*

Example 6.1.2. *Denote by R_α the reflection in the hyperplane orthogonal to α for each $\alpha \in V \setminus \{0\}$. Then we observe that for any $\alpha \in V \setminus \{0\}$, we have $M(R_\alpha) = \mathbb{F}\alpha$. Indeed, we can write for any $x \in V$:*

$$R_\alpha x = x - 2 \frac{\langle \alpha, x \rangle}{\langle \alpha, \alpha \rangle} \alpha$$

or, equivalently, $(R_\alpha - I)x = -2\frac{\langle \alpha, x \rangle}{\langle \alpha, \alpha \rangle}\alpha \in \mathbb{F}\alpha$; since $\operatorname{char}(\mathbb{F}) \neq 2$, the image of $R_\alpha - I$ is the whole of $\mathbb{F}\alpha$.

It is not hard to see that the moved and the fixed spaces are orthogonal complements of each other.

Proposition 6.1.3. *[9] For any $A \in O(V)$, we have $F(A) = (M(A))^{\perp}$.*

The dimension of the moved space satisfies the triangle inequality. It allows to relate this dimension to the word length with respect to all reflections.

Proposition 6.1.4. *[9] For $A, B \in O(V)$ the inequality $\dim M(AB) \leq \dim M(A) + \dim M(B)$ holds. The equality holds if and only if $M(AB) = M(A) \oplus M(B)$.*

This leads to the following definition by Brady and Watt:

Definition 6.1.5. *[9] We write $A \leq C$ if $\dim M(C) = \dim M(A) + \dim M(A^{-1}C)$.*

The following realization theorem by Brady and Watt is the main ingredient in their construction of normal forms.

Theorem 6.1.6. *[9] If $C \in O(V)$ and $W \subset M(C)$ is a subspace, then there exists a unique $A \in O(V)$ with $A \leq C$ and $M(A) = W$.*

Iterating this procedure, one obtains the following corollary.

Corollary 6.1.7. *[9] Let $C \in O(V)$ have $\dim M(C) = k$ and fix a maximal flag $W_1 \subset W_2 \subset \ldots \subset W_k = M(C)$ in $M(C)$. Then C factors uniquely as a product of k reflections in hyperplanes, $C = R_1 \ldots R_k$ such that $M(R_1 R_2 \ldots R_i) = W_i$ for all $i \in \{1, 2, \ldots, k\}$.*

Given this, we will define easily a norm on $O(V)$ and identify it with the word length norm with respect to all reflections in hyperplanes (in what follows, we always mean by "reflections" reflections in some hyperplanes).

Lemma 6.1.8. *For any $A, B \in O(V)$, we have $M(AB) \subset M(A) + M(B)$.*

Proof. This easy fact is implicitly proven in [9]. We know that $F(A) \cap F(B) \subset F(AB)$ (since each vector fixed by A and B remains fixed after applying AB) and $(U \cap W)^\perp = U^\perp + W^\perp$ (cf. [9]) for any two subspaces $U, W \subset V$. The claim now follows from Proposition 6.1.3. $\qquad\square$

Lemma 6.1.9. *The map $A \mapsto \dim M(A)$ defines a norm on $O(V)$, and this norm coincides with the word length norm with respect to all reflections.*

Proof. Denote by $N_\mathcal{R}$ the word length norm with respect to all reflections. First, we assume that $A \in O(V)$ can be written as a reduced word $A = R_{\alpha_1} \ldots R_{\alpha_k}$, i.e., $N_\mathcal{R}(A) = k$. Then by inductive application of Lemma 6.1.8, we obtain

$$M(A) = M(R_{\alpha_1} \ldots R_{\alpha_k}) \subset \mathbb{F}\alpha_1 + \ldots + \mathbb{F}\alpha_k,$$

thus $\dim M(A) \leq k = N_\mathcal{R}(A)$.

For the other inequality, we use Corollary 6.1.7. After choosing any maximal flag in $M(A)$, where $\dim M(A) = m$, Corollary 6.1.7 implies that there exist reflections $R_1, \ldots, R_m$ with $A = R_1 \ldots R_m$, thus $N_\mathcal{R}(A) \leq m = \dim M(A)$. This yields the claim. $\qquad\square$

The following lemma is essentially due to Carter ([15]). We prove it here since Carter proves this result for Weyl groups. Yet, the proof goes quite analogously; similar arguments can also be found in [2].

Lemma 6.1.10. *[15] Let $\alpha_1, \ldots, \alpha_k$ be vectors in V. Then the word $R_{\alpha_1} \ldots R_{\alpha_k}$ is reduced (i.e., it cannot be written as a product of fewer reflections in $O(V)$) if and only if $\alpha_1, \ldots, \alpha_k$ are linearly independent. In this case,*

$$M(R_{\alpha_1} \ldots R_{\alpha_k}) = \mathbb{F}\alpha_1 \oplus \ldots \oplus \mathbb{F}\alpha_k.$$

Proof. Write $A = R_{\alpha_1} \ldots R_{\alpha_k}$.

First, we assume this word is reduced. Then, by Lemma 6.1.9 we know $\dim M(A) = k$. On the other hand, by Lemma 6.1.8 we have

$$M(A) \subset \mathbb{F}\alpha_1 + \ldots + \mathbb{F}\alpha_k$$

thus the latter vector space is also k-dimensional and $\alpha_1, \ldots, \alpha_k$ are linearly independent. Furthermore, we note that in this case, we have

$$M(R_{\alpha_1} \ldots R_{\alpha_k}) = \mathbb{F}\alpha_1 \oplus \ldots \oplus \mathbb{F}\alpha_k.$$

Conversely, assume $\alpha_1, \ldots, \alpha_k$ are linearly independent. We proceed by induction on k. For $k = 1$, there is nothing to show. For $k = 2$, the determinant of A is $+1$ and the word $R_{\alpha_1} R_{\alpha_2}$ is either trivial or reduced; it is trivial if and only if the reflections coincide and α_1, α_2 are linearly dependent. (For this fact in the case of a general anisotropic symmetric bilinear form, cf. [46], Section 6.4).

For the induction step, assume we have shown the claim for all integers $\leq k - 1$. Then we already know by the induction hypothesis (and by the first part of the proof) that

$$M(R_{\alpha_2} \ldots R_{\alpha_k}) = \mathbb{F}\alpha_2 \oplus \ldots \oplus \mathbb{F}\alpha_k$$

Consider in $\mathbb{F}\alpha_1 \oplus \ldots \oplus \mathbb{F}\alpha_k$ a non-zero vector y orthogonal to the hyperplane $\mathbb{F}\alpha_2 \oplus \ldots \oplus \mathbb{F}\alpha_k$. Such a y can be constructed using the Gram-Schmidt procedure (without normalization). Then $\langle y, \alpha_1 \rangle \neq 0$ since otherwise y would be orthogonal to the basis $\alpha_1, \ldots, \alpha_k$ and thus vanish. Then we know that $R_{\alpha_i} y = y$ for all $i \in \{2, \ldots, k\}$ and thus

$$
\begin{aligned}
(A - I)y &= Ay - y = R_{\alpha_1} \ldots R_{\alpha_k} y - y = R_{\alpha_1} y - y \\
&= y - 2\frac{\langle y, \alpha_1 \rangle}{\langle \alpha_1, \alpha_1 \rangle}\alpha_1 - y = -2\frac{\langle y, \alpha_1 \rangle}{\langle \alpha_1, \alpha_1 \rangle}\alpha_1
\end{aligned}
$$

It follows that $\alpha_1 \in M(A)$. Now, since $\alpha_i \in M(R_{\alpha_2} \ldots R_{\alpha_k})$ for $i \in \{2, \ldots, k\}$, we can find $x_i \in V$ such that

$$R_{\alpha_2} \ldots R_{\alpha_k} x_i = x_i + \alpha_i \text{ for } i \in \{2, \ldots, k\}.$$

Apply now to both sides R_{α_1} and obtain:

$$Ax_i = x_i + \alpha_i - 2\frac{\langle x_i + \alpha_i, \alpha_1 \rangle}{\langle \alpha_1, \alpha_1 \rangle}\alpha_1.$$

Thus, $\alpha_i \in M(A)$ for $i \in \{2, \ldots, k\}$ and $\mathbb{F}\alpha_1 \oplus \ldots \oplus \mathbb{F}\alpha_k \subset M(A)$. By iterated application of Lemma 6.1.8, we know that $M(A) \subset \mathbb{F}\alpha_1 + \ldots + \mathbb{F}\alpha_k$. This yields together with Lemma 6.1.9 the claim. $\qquad\square$

Remark 6.1.11. *We have to remark that the results of this section are interesting almost only for infinite fields: Over a finite field, a symmetric bilinear form on a vector space of dimension ≥ 3 is necessarily isotropic (cf. [46], Exercise 6 of Section 6.4).*

6.2 Factorability Structure on $O(V)$

Again, let V be an n-dimensional vector space over a field $\mathbb{F}$, $\mathrm{char}(\mathbb{F}) \neq 2$, with an anisotropic symmetric bilinear form $\langle -, - \rangle$. Fix for the rest of the section a maximal flag $\mathcal{V}$ in V:

$$\{0\} = V_0 \subset V_1 \subset \ldots \subset V_n = V$$

with $\dim V_i = i$.

We now define a factorization map on $O(V)$ with respect to the norm of Lemma 6.1.9. For any non-trivial $A \in O(V)$, the maximal flag $\mathcal{V}$ in V induces (after forgetting equal subspaces) a maximal flag in $M(A)$: There exist $1 \leq i_1 < i_2 < \ldots < i_m = n$ (where $m = \dim M(A)$) such that for any $1 \leq k \leq m$, we have $\dim M(A) \cap V_{i_k} = k$. Then, by Corollary 6.1.7, there exist up to scaling unique vectors $\alpha_1, \alpha_2, \ldots, \alpha_m$ such that $A = R_{\alpha_1} \ldots R_{\alpha_m}$ and $M(R_{\alpha_1} \ldots R_{\alpha_k}) = M(A) \cap V_{i_k}$ for all $1 \leq k \leq m$. Set

$$\eta(A) = (R_{\alpha_1} \ldots R_{\alpha_{m-1}}, R_{\alpha_m})$$

We will show:

Proposition 6.2.1 (Bödigheimer, O.)**.** *The map η defined above provides a factorability structure on $(O(V), N_{\mathcal{R}})$.*

Proof. By definition, (F1) and (F3) are obviously satisfied. Furthermore, by definition $\dim M(R_{\alpha_1} \ldots R_{\alpha_{m-1}}) = m - 1$ and by Lemma 6.1.9 we observe that (F2) is satisfied. Thus, η is a factorization map. Next, we show that the axioms (F4) and (F5) are fulfilled. We use the reformulation of the Lemma 2.1.6. Let A be any element of $O(V)$ and let $B = R_\beta$ be a reflection in the hyperplane orthogonal to β.

As in Lemma 2.1.6, we have to consider the pairs

$$(R_{\alpha_m}, R_\beta) \text{ and } (R_{\alpha_1} \ldots R_{\alpha_{m-1}}, \overline{\eta}(R_{\alpha_m} R_\beta)).$$

As mentioned before, the first pair is geodesic if and only if β is not a multiple of α_m. If so, we have by definition to choose for the second pair a $j \in \{1, \ldots, n\}$ such that

$$V_j \cap (\mathbb{F}\alpha_m \oplus \mathbb{F}\beta) = \mathbb{F}\gamma.$$

Then $\overline{\eta}(R_{\alpha_m} R_\beta) = R_\gamma$.

By Remark 2.1.5, if (A, B) is a geodesic pair, both other pairs in question are automatically geodesic.

We prove the other implication: Assume that the pairs (R_{α_m}, R_β) and

$$(R_{\alpha_1} \ldots R_{\alpha_{m-1}}, R_\gamma)$$

are geodesic. Equivalently, we know that α_m is not a multiple of β and that the vectors $\alpha_1, \ldots, \alpha_{m-1}, \gamma$ are linearly independent by Lemma 6.1.10. We have to show that $\alpha_1, \ldots, \alpha_m, \beta$ are linearly independent. Suppose not. Since the vectors $\alpha_1, \ldots, \alpha_m$ are linearly independent, we have $\beta \in \mathbb{F}\alpha_1 \oplus \ldots \oplus \mathbb{F}\alpha_m = M(A)$. Thus, $\gamma \in M(A)$ since it is a linear combination of β and α_m. On the other hand, by assumption $\gamma \notin \mathbb{F}\alpha_1 \oplus \ldots \oplus \mathbb{F}\alpha_{m-1} = M(A) \cap V_{i_{m-1}}$ and we conclude that $\gamma \notin V_{i_{m-1}}$. Since γ is contained in $M(A) \cap V_j$, this subspace cannot be contained in $M(A) \cap V_{i_{m-1}}$, thus by definition $M(A) \cap V_j = M(A) \cap V_{i_m} = M(A)$, i.e. $M(A) \subset V_j$. But then $\mathbb{F}\alpha_m \oplus \mathbb{F}\beta \subset V_j$, contradicting the choice of j. This completes the proof of the first condition in the Lemma 2.1.6.

Thus, it remains to show that the last condition of Lemma 2.1.6 is satisfied. Assume (A, B) is a geodesic pair as above. Let $C = \overline{\eta}(AB)$. Since the axiom (F2) is fulfilled, we

know that $C \leq AB$ in the sense of Definition 6.1.5. Furthermore, $R_{\alpha_1} \dots R_{\alpha_{m-1}} R_\gamma \leq AB$ since the norm of $R_{\alpha_1} \dots R_{\alpha_{m-1}} R_\gamma$ is m by above consideration. In order to complete the proof of the proposition, it is enough to show that $M(C) = M(R_{\alpha_1} \dots R_{\alpha_{m-1}} R_\gamma)$. Indeed, the last one is by Lemma 6.1.10 equal to $\mathbb{F}\alpha_1 \oplus \dots \oplus \mathbb{F}\alpha_{m-1} \oplus \mathbb{F}\gamma$ and thus a subspace of $\mathbb{F}\alpha_1 \oplus \dots \oplus \mathbb{F}\alpha_m \oplus \mathbb{F}\beta = M(AB)$ (the last equality holds, again, due to 6.1.10). By the uniqueness part of Theorem 6.1.6, we're done then.

By definition, $M(C)$ is

$$V_t \cap M(AB) = V_t \cap (\mathbb{F}\alpha_1 \oplus \dots \oplus \mathbb{F}\alpha_m \oplus \mathbb{F}\beta).$$

where $t \in \{1, \dots, n\}$ has to fulfill $\dim(V_t \cap M(AB)) = m$. Since $M(R_{\alpha_1} \dots R_{\alpha_{m-1}} R_\gamma) = \mathbb{F}\alpha_1 \oplus \dots \oplus \mathbb{F}\alpha_{m-1} \oplus \mathbb{F}\gamma$ is also m - dimensional, it is enough to show that one of those two subspaces of V is contained in the other one. Write for short $W := \mathbb{F}\alpha_1 \oplus \dots \oplus \mathbb{F}\alpha_{m-1} = M(A) \cap V_{i_{m-1}}$. We have to consider several cases.

1. Let $j \leq i_{m-1}$. We consider

$$V_{i_{m-1}} \cap M(AB) = V_{i_{m-1}} \cap (W \oplus \mathbb{F}\alpha_m \oplus \mathbb{F}\beta).$$

   This vector space is at most m-dimensional since it is a subspace of an $m + 1$-dimensional vector space $W \oplus \mathbb{F}\alpha_m \oplus \mathbb{F}\beta$ and does not contain α_m. On the other hand, by definition $W \subset V_{i_{m-1}}$ thus $W \subset V_{i_{m-1}} \cap (W \oplus \mathbb{F}\alpha_m \oplus \mathbb{F}\beta)$ and, since $\gamma \in V_j \subset V_{i_{m-1}}$, we have an m-dimensional subspace

$$W \oplus \mathbb{F}\gamma \subset V_{i_{m-1}} \cap (W \oplus \mathbb{F}\alpha_m \oplus \mathbb{F}\beta).$$

   Thus we have $V_t \cap M(AB) = V_{i_{m-1}} \cap M(AB) = W \oplus \mathbb{F}\gamma$ and we're done in this case.

2. Let now $i_{m-1} \leq j$. We consider

$$V_j \cap M(AB) = V_j \cap (W \oplus \mathbb{F}\alpha_m \oplus \mathbb{F}\beta)$$

   First, we observe again that since $\mathbb{F}\alpha_m \oplus \mathbb{F}\beta$ is not contained in V_j (by definition of j), this vector space is at most m-dimensional. Since $W \subset V_{i_{m-1}} \subset V_j$, we have $W \subset V_j \cap (W \oplus \mathbb{F}\alpha_m \oplus \mathbb{F}\beta)$. Furthermore,

$$\gamma \in V_j \cap (\mathbb{F}\alpha_m \oplus \mathbb{F}\beta) \subset V_j \cap (W \oplus \mathbb{F}\alpha_m \oplus \mathbb{F}\beta).$$

   Thus we conclude as in the first case that $V_t \cap M(AB) = V_j \cap M(AB) = W \oplus \mathbb{F}\gamma$ and we're done.

$\square$

6.3 Geometric Interpretation

We are going to show now that the geometric interpretation of the factorability structure on $O(V)$ given by C.-F.Bödigheimer coincides with the description given earlier.

Fix a flag $(V_i)_{1 \leq i \leq n}$ in V determining a factorability structure on $O(V)$ as in Proposition 6.2.1. Fix furthermore an orthogonal basis $e_1, e_2, \ldots, e_n$ corresponding to this flag, i.e., we have $V_i = \mathbb{F}e_1 \oplus \ldots \oplus \mathbb{F}e_i$. Consider $A \in O(V)$. Then there is a $1 \leq k \leq n$ such that $M(A) \subset V_k$ but $M(A) \not\subset V_{k-1}$, where $V_0 := \{0\}$. Since $M(A)$ is the orthogonal complement of $F(A)$ by Proposition 6.1.3 and $\mathbb{F}e_{k+1} \oplus \ldots \oplus \mathbb{F}e_n$ is the orthogonal complement of V_k, we have $\mathbb{F}e_{k+1} \oplus \ldots \oplus \mathbb{F}e_n \subset F(A)$. So, in particular, $e_{k+1}, \ldots, e_n$ are fixed by A. On the other hand, if e_k was fixed by A, we would have $\mathbb{F}e_k \oplus \ldots \oplus \mathbb{F}e_n \subset F(A)$ and thus $M(A) \subset V_{k-1}$, contradicting our assumption. From this, k can be described as the largest index such that e_k is not fixed by A.

We claim that $\eta'(A)$ is the reflection at the hyperplane orthogonal to $A^{-1}e_k - e_k$, so that multiplication with this reflection turns e_k into an additional fixed basis vector. We make this fact precise in the following Proposition. The situation is illustrated in the picture below.
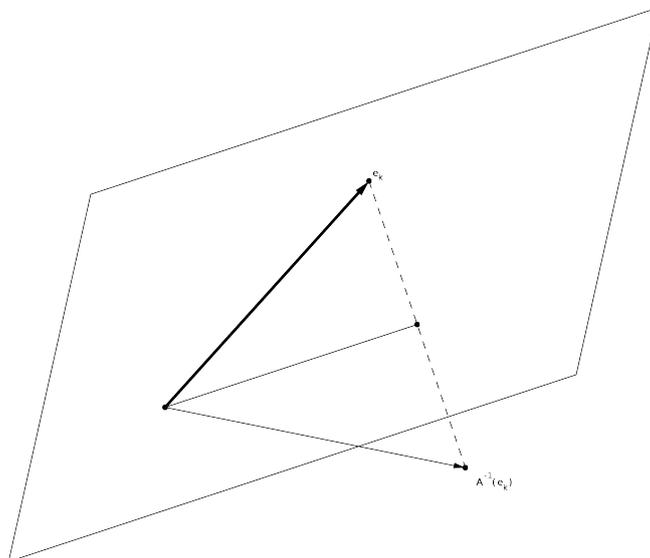
Proposition 6.3.1. *Let A be in $O(V)$, and k, as above, the largest index such that e_k is not fixed by A. Then $\eta'(A) = R_\alpha$ with $\alpha = A^{-1}e_k - e_k$.*

Proof. We will prove that $M(AR_\alpha) = M(A) \cap V_{k-1}$. This will yield the claim, since then $\dim M(AR_\alpha) = \dim M(A) - 1$ and we can use Theorem 6.1.6 and the definition of the factorability structure on $O(V)$.

First, we check that $AR_\alpha(e_k) = e_k$. Note that α is not 0 since $A\alpha = e_k - Ae_k \neq 0$ by considerations above. We have by definition

$$R_\alpha(e_k) = e_k - 2\frac{\langle e_k, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha.$$

Note that

$$\langle \alpha, \alpha \rangle = \langle A^{-1}e_k - e_k, A^{-1}e_k - e_k \rangle = 2\langle e_k, e_k \rangle - 2\langle e_k, A^{-1}e_k \rangle = -2\langle e_k, \alpha \rangle$$

since A^{-1} is also orthogonal. Thus $R_\alpha(e_k) = e_k + \alpha = A^{-1}e_k$ and $AR_\alpha(e_k) = e_k$.

Next, we observe that $\alpha \in M(A)$. Indeed, we can write $\alpha = A(-A^{-1}e_k) - (-A^{-1}e_k)$. In particular, $\langle \alpha, e_j \rangle = 0$ for $j \geq k + 1$ and $AR_\alpha(e_j) = e_j$ for $j \geq k + 1$.

Thus, we conclude as above that $M(AR_\alpha) \subset V_{k-1}$. On the other hand, by Lemma 6.1.8, we have $M(AR_\alpha) \subset M(A) + M(R_\alpha) = M(A) + \mathbb{F}\alpha = M(A)$, since $\alpha \in M(A)$. So $M(AR_\alpha) \subset M(A) \cap V_{k-1}$.

Last, by Lemma 6.1.4, we have $\dim M(A) \leq \dim M(AR_\alpha) + \dim M(R_\alpha)$, so

$$\dim M(AR_\alpha) \geq \dim M(A) - 1 = \dim(M(A) \cap V_{k-1}).$$

This yields $M(AR_\alpha) = M(A) \cap V_{k-1}$ and also the claim of the Proposition.

$\square$

6.4 Factorability Structures on Some Coxeter Groups

We consider the Coxeter groups of type $A_{n-1} = \mathfrak{S}_n$ and B_n. Unfortunately, the Coxeter groups of the B-series are usually denoted by the same letter as usual braid groups, which are Artin groups corresponding to the A-series. We hope that this does not cause confusion. Recall that the Coxeter group B_n may be identified with the semi-direct product $(\mathbb{Z}/2)^n \rtimes \mathfrak{S}_n$, where the symmetric group acts on $(\mathbb{Z}/2)^n$ by permutations.

Our goal is to show that the Coxeter groups of type A and B inherit the factorability structure from the orthogonal group, and for symmetric groups, this factorability structure coincides with the one defined by Visy (cf. [60]). Recall that we have $A_{n-1} \subset B_n \subset O(n)$, where $O(n)$ denotes now the usual orthogonal group of $\mathbb{R}^n$ with standard scalar product. It is well-known that: (see e.g. [45])

Lemma 6.4.1. *The Coxeter groups of type A_{n-1} have a root system consisting of*

$$T(A_{n-1}) \cup -T(A_{n-1}) \text{ with } T(A_n) = \{e_i - e_j | 1 \leq i < j \leq n\}.$$

The Coxeter groups of type B_n have a root system consisting of $T(B_n) \cup -T(B_n)$ with

$$T(B_n) = \{e_i - e_j | 1 \leq i < j \leq n\} \cup \{e_i + e_j | 1 \leq i < j \leq n\} \cup \{e_i | 1 \leq i \leq n\}$$

We are going to show the following proposition.

Proposition 6.4.2. *Let $A_{n-1} \subset B_n \subset O(n)$ be the standard inclusions and let η be the factorability structure on $O(n)$ given by Proposition 6.2.1 with respect to the flag $V_k = \oplus_{1 \leq i \leq k} \mathbb{R}e_i$. Then η restricts to B_n and A_{n-1} (with the generating set of all reflections in B_n and A_{n-1}, respectively), i.e., for any $x \in B_n$, the element $\eta(x)$ lies in the subset $B_n \times B_n$ of $O(n) \times O(n)$, and similarly for A_{n-1}.*

Proof. Write G for B_n or A_{n-1}.

Note that the reflections in G are exactly the reflections R_α with $\alpha \in T(G)$. Suppose we have shown that for $\alpha \neq \beta \in T(G)$ we have $\bar{\eta}(R_\alpha R_\beta) = R_\gamma$ with $\gamma \in T(G)$. Then we have also $\eta'(R_\alpha R_\beta) \in G$ since G is a group. For an arbitrary element x of G, choose any geodesic (in G) word in R_α's representing x with $\alpha \in T(G)$. Then we know by Theorem 2.2.4 that we either obtain a normal form in $O(n)$ by applying f_{D_n} to this word or we would decrease the norm. Note that under our assumptions, any f_i makes out of a word in R_α's with $\alpha \in T(G)$ again such a word. In particular, the norm can't drop since we started with a geodesic word in G, and the normal form lives also in G.

So we only need to show that for $\alpha \neq \beta \in T(G)$ we have $\bar{\eta}(R_\alpha R_\beta) = R_\gamma$ with $\gamma \in T(G)$. For doing so, we have to distinguish several cases. Recall that for $\alpha \neq \beta$, we have $M(R_\alpha R_\beta) = \mathbb{R}\alpha \oplus \mathbb{R}\beta$ and that γ as above is given by $V_s \cap (\mathbb{R}\alpha \oplus \mathbb{R}\beta) = \mathbb{R}\gamma$ for s defined by $\mathbb{R}\alpha \oplus \mathbb{R}\beta \subset V_{s+1}$ and $\mathbb{R}\alpha \oplus \mathbb{R}\beta \nsubseteq V_s$. Observe that it's enough to find any s such that the intersection as above is one-dimensional. We will use this below.

We start with $G = A_{n-1}$. Consider $\alpha = e_i - e_j$, $\beta = e_k - e_l$, $i < j$, $k < l$.

Case 1: If $j < l$, we have $V_{l-1} \cap (\mathbb{R}\alpha \oplus \mathbb{R}\beta) = \mathbb{R}(e_i - e_j)$.

Case 2: If $l < j$, we have symmetrically $V_{j-1} \cap (\mathbb{R}\alpha \oplus \mathbb{R}\beta) = \mathbb{R}(e_k - e_l)$.

Case 3: If $l = j$, note that $\alpha - \beta = e_i - e_k$ is again a root. Thus $V_{l-1} \cap (\mathbb{R}\alpha \oplus \mathbb{R}\beta) = \mathbb{R}(e_i - e_k)$ as desired.

So we see that A_{n-1} is a factorable subgroup of $O(n)$. Observe we need not to check these cases again for B_n.

Now we examine $G = B_n$.

Case 1: Let $\alpha = e_i + e_j$, $\beta = e_k + e_l$, $i < j$, $k < l$

Case 1.1: If $j < l$, we have $V_{l-1} \cap (\mathbb{R}\alpha \oplus \mathbb{R}\beta) = \mathbb{R}(e_i + e_j)$.

Case 1.2: If $l < j$, we have symmetrically $V_{j-1} \cap (\mathbb{R}\alpha \oplus \mathbb{R}\beta) = \mathbb{R}(e_k + e_l)$.

Case 1.3: If $l = j$, note that $\alpha - \beta = e_i - e_k$ is again a root. Thus $V_{l-1} \cap (\mathbb{R}\alpha \oplus \mathbb{R}\beta) = \mathbb{R}(e_i - e_k)$ as desired.

Case 2: Let $\alpha = e_i - e_j$, $\beta = e_k + e_l$, $i < j$, $k < l$.

Case 2.1: If $j < l$, we have $V_{l-1} \cap (\mathbb{R}\alpha \oplus \mathbb{R}\beta) = \mathbb{R}(e_i - e_j)$.

Case 2.2: If $l < j$, we have $V_{j-1} \cap (\mathbb{R}\alpha \oplus \mathbb{R}\beta) = \mathbb{R}(e_k + e_l)$.

Case 2.3: If $l = j$ and $i \neq k$, note that $\alpha + \beta = e_i + e_k$ is again a root. Thus

$$V_{l-1} \cap (\mathbb{R}\alpha \oplus \mathbb{R}\beta) = \mathbb{R}(e_i + e_k)$$

as desired.

Case 2.4: If $l = j$ and $i = k$, note that $\frac{1}{2}(\alpha + \beta) = e_i$ is a root. Thus we have $V_{l-1} \cap (\mathbb{R}\alpha \oplus \mathbb{R}\beta) = \mathbb{R}e_i$ as desired.

Case 3: Let $\alpha = e_i \pm e_j$, $\beta = e_k$, $i < j$.

    Case 3.1: If $j < k$, we have $V_{k-1} \cap (\mathbb{R}\alpha \oplus \mathbb{R}\beta) = \mathbb{R}(e_i \pm e_j)$.

    Case 3.2: If $k < j$, we have similarly $V_{j-1} \cap (\mathbb{R}\alpha \oplus \mathbb{R}\beta) = \mathbb{R}e_k$.

    Case 3.3: If $k = j$, note that $\alpha \mp \beta = e_i$ is again a root. Thus

$$V_{k-1} \cap (\mathbb{R}\alpha \oplus \mathbb{R}\beta) = \mathbb{R}e_i$$

      as desired.

Case 4: Last, assume $\alpha = e_i$, $\beta = e_j$. In this case, the claim is obvious. This completes the proof.

$\square$

Remark 6.4.3. *The factorability structure on $\mathfrak{S}_n$ obtained in Proposition 6.4.2 is exactly the Visy factorability structure as described in Example 2.1.8.*

Proof. We will check that the restriction of the factorability structure on $O(n)$ and the Visy factorability structure coincide when evaluated at a product of transpositions. This is enough since a factorability structure is determined by the corresponding local factorability structure by a result of M. Rodenhausen (cf. Theorem 2.3.2), i.e., the factorability structure is determined by its values on the products of two generators.

First, observe $R_{e_i - e_j}$ is the image of (ij) under the standard inclusion into $O(n)$; the factorability structure of 6.4.2 can be characterized as follows. Assume $i < j$, $k < l$.

$$\eta((ij)(kl)) = \begin{cases} ((ij),(kl)), j < l \\ ((kl),(lj)), l < j, i = k \\ ((kl),(kj)), l < j, i = l \\ ((kl),(ij)), l < j, i \notin \{k,l\} \\ ((ik),(il)), l = j, i \neq k \end{cases}$$

We compare this with the Visy factorability structure:

    Assume again $i < j$, $k < l$ and consider $(ij)(kl)$.

1. If $j < l$, then l is the largest non-fixed point of $(ij)(kl)$, and

$$((ij)(kl))^{-1}(l) = (kl)(ij)(l) = k,$$

    since $i < j < l$. Thus $\eta'((ij)(kl)) = (kl)$ in this case.

2. If $l < j$ and $i \notin \{k,l\}$, then $(ij)(kl) = (kl)(ij)$ and by the previous case $\eta'((ij)(kl)) = (ij)$.

3. If $l < j$ and $i = k$, then $(kj)(kl) = (klj)$, so $(klj)^{-1}(j) = l$ and $\eta'((ij)(kl)) = (lj)$.

4. If $l < j$ and $i = l$, then $(lj)(kl) = (kjl)$, so $(kjl)^{-1}(j) = k$ and $\eta'((ij)(kl)) = (kj)$.

5. If $l = j$ and $i \neq k$, then $(il)(kl) = (ilk)$, so $(ilk)^{-1}(l) = i$ and $\eta'((ij)(kl)) = (il)$.

This completes the proof. $\square$

7 Rewriting Systems out of Factorability Structures

We want to deal with the question when a factorability structure on a monoid provides a complete rewriting system (as defined in Section 3.1) for this monoid. This question was raised by A. Heß. We always obtain a rewriting system with exactly one irreducible element in each equivalence class, but we will show that it is not necessarily noetherian. We will yet exhibit several cases where the given rewriting system is noetherian.

7.1 Rewriting System of a Factorable Monoid

First, we are going to make precise which rewriting system is going to be associated to a factorability structure on a monoid. The choice is quite self-evident.

Lemma 7.1.1 (Heß, O.)**.** *Let $(M, \mathcal{E}, \eta)$ be a factorable monoid. Then the rewriting rules*

$$(x, y) \to \eta(xy) \text{ for } x, y \in \mathcal{E} \text{ if } (x, y) \text{ is unstable}$$

define a strongly minimal rewriting system on M with exactly one irreducible element in each equivalence class. (here, if $\overline{xy} = 1$, we interpret the rewriting rule as $(x, y) \to xy$).

Proof. First, by Proposition 2.3.2 of M. Rodenhausen, we know that this rewriting system defines exactly the monoid M we started with. Since the right-hand side of each rewriting rule is always a stable pair or a single element of $\mathcal{E}$ or 1, we know that right sides of our rewriting rules are irreducible. Furthermore, the elements of $\mathcal{E}$ are irreducible, and left-hand sides are irreducible if we remove the rule containing them. Thus this rewriting system is strongly minimal.

Furthermore, each element of the monoid can be brought into its normal form with the normal form procedure described in Section 2.3, which is obviously a chain of applications of the rewriting rules above. Furthermore, we observed in Lemma 2.3.3 that the obtained normal form is totally stable, which translates exactly into irreducible in the language of rewriting systems. This implies there is exactly one irreducible element in each equivalence class of words in the free monoid on $\mathcal{E}$ (under the equivalence relation generated by above rules). □

Unfortunately, this rewriting system is not always noetherian, even if $\mathcal{E}$ is finite and the resulting monoid is right-cancellative. We present now an example of a factorable monoid where the associated rewriting system is not noetherian. We prove the factorability by exhibiting a local factorability structure and using then the criterion 2.3.2.

Since the proof is quite technical, we first show how the example was constructed. We want to find a cycle in the system of rewritings when $(\varphi_3\varphi_2\varphi_1\varphi_2)^N$ is applied. (This can be checked to be the only candidate to produce a cycle when applied to a 4-tuple.) One can construct a monoid with a cycle of length $N = 1$, but naive attempts yield then a not right-cancellative monoid. So we will force the cycle to have length 2. Thus, we have necessarily rewritings of the following form:
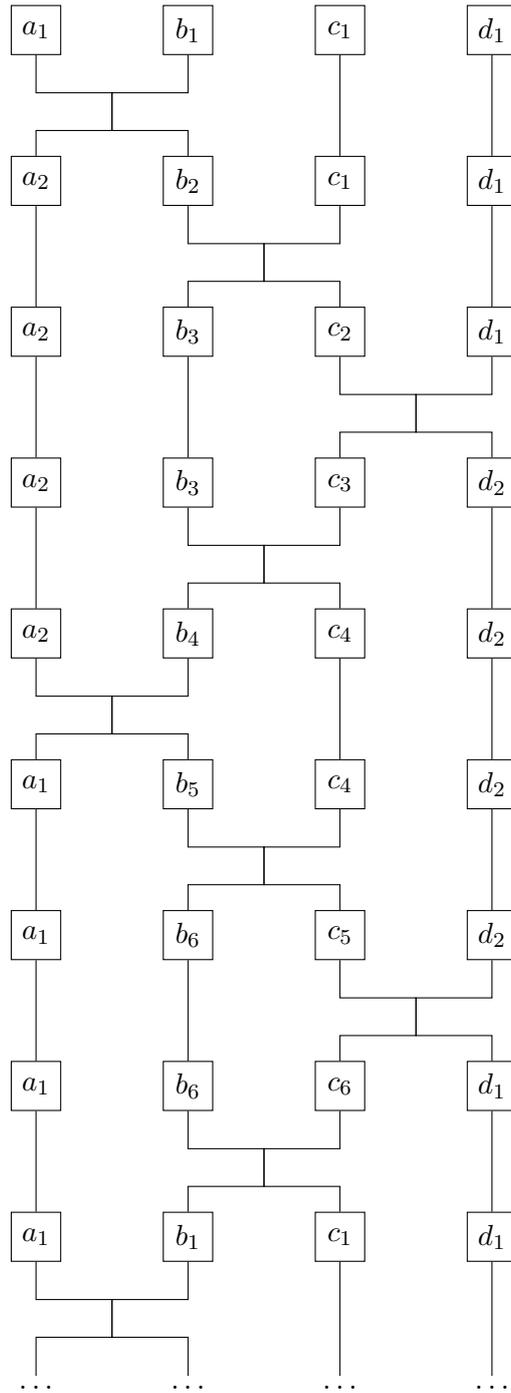
Furthermore, one can see that the pairs (a_2, b_3) and (a_1, b_6) should not be geodesic since otherwise, the rewritings have to stabilize. The further rewriting rules arose during the proof. This should justify the definition in the following proposition.

Proposition 7.1.2. *Let $\mathcal{E}$ be the following set:*

$$\{a_1, a_2, b_1, b_2, b_3, b_4, b_5, b_6, c_1, c_2, c_3, c_4, c_5, c_6, d_1, d_2, e_2, e_3, f_2, f_3, g_2, g_3, h_2, h_3, i, j, k\}$$

Define a function $\varphi\colon \mathcal{E}^+ \times \mathcal{E}^+ \to \mathcal{E}^+ \times \mathcal{E}^+$ as follows:

$$
\begin{aligned}
\varphi(a_1, b_1) &= (a_2, b_2) \\
\varphi(b_2, c_1) &= (b_3, c_2) \\
\varphi(c_2, d_1) &= (c_3, d_2) \\
\varphi(b_3, c_3) &= (b_4, c_4) \\
\varphi(a_2, b_4) &= (a_1, b_5) \\
\varphi(b_5, c_4) &= (b_6, c_5) \\
\varphi(c_5, d_2) &= (c_6, d_1) \\
\varphi(b_6, c_6) &= (b_1, c_1) \\
\varphi(a_2, b_3) &= (1, e_2) \\
\varphi(a_1, b_6) &= (1, e_3) \\
\varphi(e_2, c_2) &= (f_2, g_2) \\
\varphi(e_3, c_5) &= (f_3, g_3) \\
\varphi(e_2, c_3) &= (f_3, g_3) \\
\varphi(e_3, c_6) &= (f_2, g_2) \\
\varphi(g_2, d_1) &= (h_2, i) \\
\varphi(g_3, d_2) &= (h_3, i) \\
\varphi(f_2, h_2) &= (j, k) \\
\varphi(f_3, h_3) &= (j, k) \\
\varphi(s, 1) &= (1, s) \textit{ for all } s \in \mathcal{E}^+
\end{aligned}
$$

and $\varphi(s, t) = (s, t)$ if (s, t) is not in the list above.

This function is a local factorability structure in the sense of Definition 2.3.1. The associated rewriting system is not noetherian. Furthermore, the monoid M defined by this local factorability structure is right cancellative.

Proof. The map φ satisfies by definition $\varphi(x, 1) = (1, x)$ for all $x \in \mathcal{E}^+$ and also $\varphi^2 = \varphi$.

Now we are going to check the fourth condition of the Definition 2.3.1, the stability for triples condition. We will consider several cases. Note that the stability for triples condition is automatically satisfied if the triple we start with is totally stable. Moreover, we are done as soon as the triple contains a 1; thus, we do not have to consider triples already containing 1 and we are also immediately done with triples of the form (a_2, b_3, t) or (a_1, b_6, t) for all $t \in \mathcal{E}$. To make the steps more transparent, we will use graphical presentation.

Case 1: Here, we start with the triple $(\mathbf{a_1}, \mathbf{b_1}, \mathbf{t})$ for some $t \in \mathcal{E}$. Observe that we are done after applying φ_2 unless $t = c_1$, so we assume this from the second step on.

|		
a_1	b_1	t
a_2	b_2	t
a_2	b_3	c_2
1	e_2	c_2
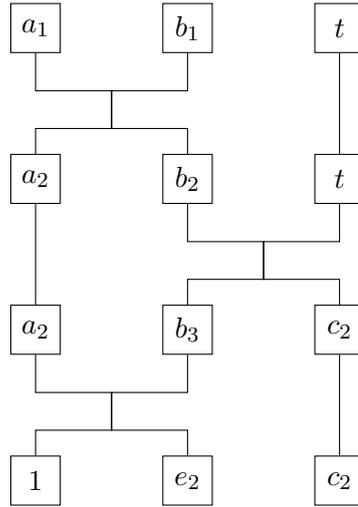
Thus, in this case, application of $\varphi_2\varphi_1\varphi_2$ yields a 1.

Case 2: We start with the triple $(\mathbf{b_2}, \mathbf{c_1}, \mathbf{t})$ for some $t \in \mathcal{E}$. Here, we are done after applying φ_2 unless $t = d_1$, so we assume this from the second step on.

b_2	c_1	t
b_3	c_2	t
b_3	c_3	d_2
b_4	c_4	d_2
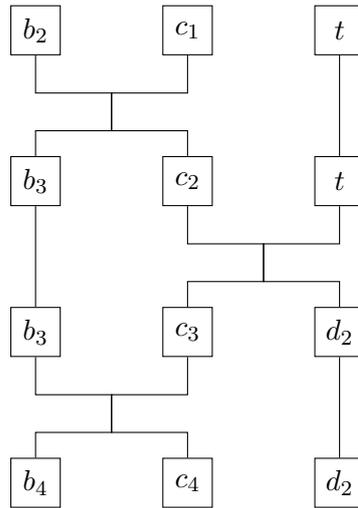
Since the pair (c_4, d_2) is stable, the resulting triple is totally stable.

Case 3: We start with the triple $(\mathbf{c_2}, \mathbf{d_1}, \mathbf{t})$ for some $t \in \mathcal{E}$. Here, we are done after applying φ_2 since all pairs of the form (d_2, t) are stable.

c_2	d_1	t
c_3	d_2	t
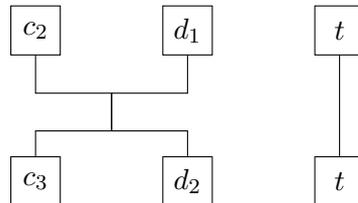
Case 4: We start with the triple $(\mathbf{b_3}, \mathbf{c_3}, \mathbf{t})$ for some $t \in \mathcal{E}$. Here, we are done after applying φ_2 since all pairs of the form (c_4, t) are stable.
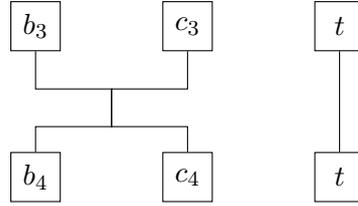
Case 5: Here, we start with the triple $(\mathbf{a_2}, \mathbf{b_4}, \mathbf{t})$ for some $t \in \mathcal{E}$. Observe that we are done after applying φ_2 unless $t = c_4$, so we assume this from the second step on.
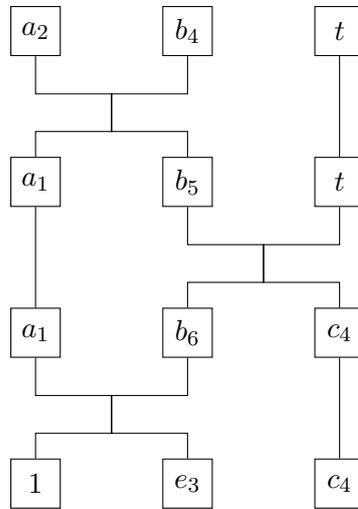
Thus, in this case, application of $\varphi_2 \varphi_1 \varphi_2$ yields a 1.

Case 6: We start with the triple $(\mathbf{b_5}, \mathbf{c_4}, \mathbf{t})$ for some $t \in \mathcal{E}$. Here, we are done after applying φ_2 unless $t = d_2$, so we assume this from the second step on.
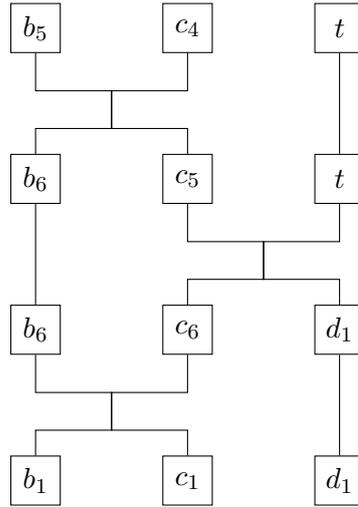
Since the pair (c_1, d_1) is stable, the resulting triple is totally stable.

Case 7: We start with the triple $(\mathbf{c_5}, \mathbf{d_2}, \mathbf{t})$ for some $t \in \mathcal{E}$. Here, we are done after applying φ_2 since all pairs of the form (d_1, t) are stable.
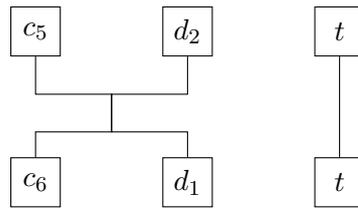
Case 8: We start with the triple $(\mathbf{b_6}, \mathbf{c_6}, \mathbf{t})$ for some $t \in \mathcal{E}$. Here, we are done after applying φ_2 since all pairs of the form (c_1, t) are stable.
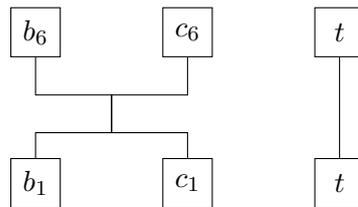

Case 9: We start with the triple $(\mathbf{e_2}, \mathbf{c_2}, \mathbf{t})$ for some $t \in \mathcal{E}$. Here, we are done after applying φ_2 unless $t = d_1$, so we assume this from the second step on.
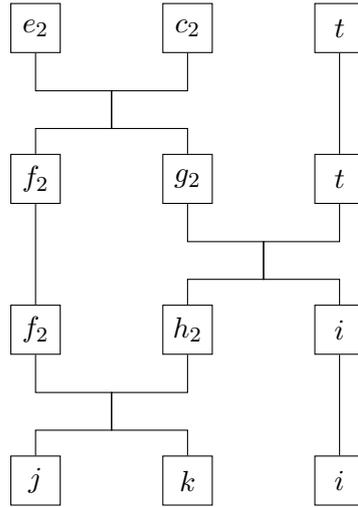
Since the pair (k, i) is stable, the resulting triple is totally stable.

Case 10: We start with the triple $(\mathbf{e_3}, \mathbf{c_5}, \mathbf{t})$ for some $t \in \mathcal{E}$. Here, we are done after applying φ_2 unless $t = d_2$, so we assume this from the second step on.
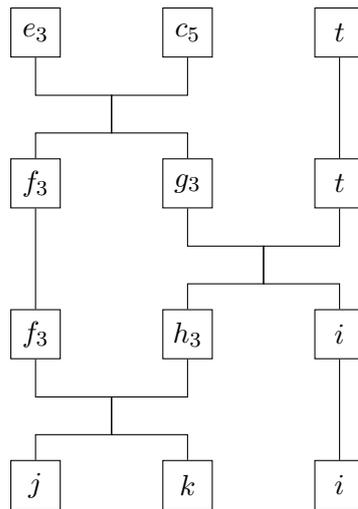
Since the pair (k, i) is stable, the resulting triple is totally stable.

Case 11: We start with the triple $(\mathbf{e_2}, \mathbf{c_3}, \mathbf{t})$ for some $t \in \mathcal{E}$. Here, we are done after applying φ_2 unless $t = d_2$, so we assume this from the second step on.

$$
\begin{array}{ccc}
\boxed{e_2} & \boxed{c_3} & \boxed{t} \\
& & \\
\boxed{f_3} & \boxed{g_3} & \boxed{t} \\
& & \\
\boxed{f_3} & \boxed{h_3} & \boxed{i} \\
& & \\
\boxed{j} & \boxed{k} & \boxed{i}
\end{array}
$$
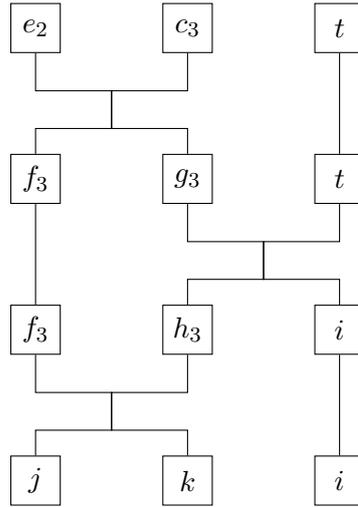
Since the pair (k, i) is stable, the resulting triple is totally stable.

Case 12: We start with the triple $(\mathbf{e_3}, \mathbf{c_6}, \mathbf{t})$ for some $t \in \mathcal{E}$. Here, we are done after applying φ_2 unless $t = d_1$, so we assume this from the second step on.

$$
\begin{array}{ccc}
\boxed{e_3} & \boxed{c_6} & \boxed{t} \\
& & \\
\boxed{f_2} & \boxed{g_2} & \boxed{t} \\
& & \\
\boxed{f_2} & \boxed{h_2} & \boxed{i} \\
& & \\
\boxed{j} & \boxed{k} & \boxed{i}
\end{array}
$$
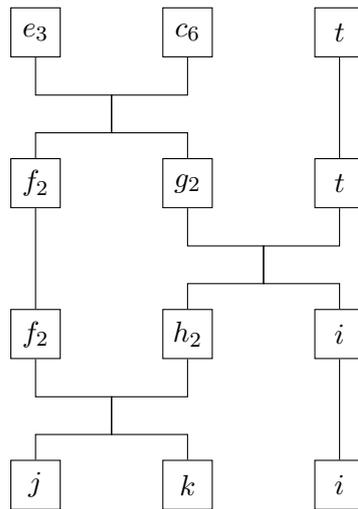
Since the pair (k, i) is stable, the resulting triple is totally stable.

Case 13: We start with the triple $(\mathbf{g_2}, \mathbf{d_1}, \mathbf{t})$ for some $t \in \mathcal{E}$. Here, we are done after applying φ_2 since all pairs of the form (i, t) are stable.

$$
\begin{array}{ccc}
\boxed{g_2} & \boxed{d_1} & \boxed{t} \\
& & \\
\boxed{h_2} & \boxed{i} & \boxed{t}
\end{array}
$$

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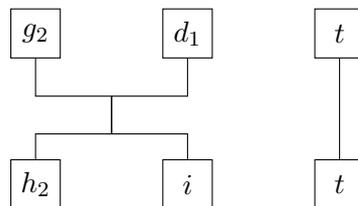
Case 14: We start with the triple $(\mathbf{g_3}, \mathbf{d_2}, \mathbf{t})$ for some $t \in \mathcal{E}$. Here, we are done after applying φ_2 since all pairs of the form (i, t) are stable.

$$
\begin{array}{ccc}
\boxed{g_3} & \boxed{d_2} & \boxed{t} \\
& & \\
\boxed{h_3} \quad \boxed{i} & & \boxed{t}
\end{array}
$$
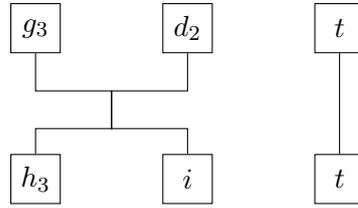
Case 15: We start with the triple $(\mathbf{f_n}, \mathbf{h_n}, \mathbf{t})$ for some $t \in \mathcal{E}$ and $n \in \{2, 3\}$. Here, we are done after applying φ_2 since all pairs of the form (k, t) are stable.

$$
\begin{array}{ccc}
\boxed{f_n} & \boxed{h_n} & \boxed{t} \\
& & \\
\boxed{j} \quad \boxed{k} & & \boxed{t}
\end{array}
$$
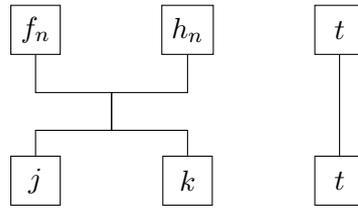
From now on, we may assume that the first pair is stable, so we only have to show that the application of $\varphi_2 \varphi_1$ to such a triple yields either a 1 or a totally stable triple.

Case 16: We consider the triple $(\mathbf{t}, \mathbf{a_1}, \mathbf{b_1})$ for some $t \in \mathcal{E}$. Here, we are done after φ_1 since all pairs of the form (t, a_2) are stable.

$$
\begin{array}{ccc}
\boxed{t} & \boxed{a_1} & \boxed{b_1} \\
& & \\
\boxed{t} & \boxed{a_2} & \boxed{b_2}
\end{array}
$$
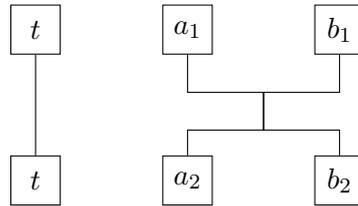
Case 17: We consider the triple $(\mathbf{t}, \mathbf{b_2}, \mathbf{c_1})$ for some $t \in \mathcal{E}$. Here, we are done after applying φ_1 unless $t = a_2$, so we assume this in the second step.

$$
\begin{array}{ccc}
\boxed{t} & \boxed{b_2} & \boxed{c_1} \\
& & \\
\boxed{t} & \boxed{b_3} & \boxed{c_2} \\
& & \\
\boxed{1} & \boxed{e_2} & \boxed{c_2}
\end{array}
$$
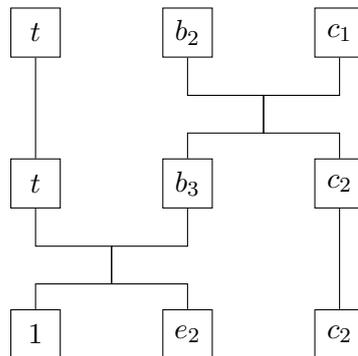
We are done in this case since this generates a 1.

Case 18: We consider the triple $(\mathbf{t}, \mathbf{c_2}, \mathbf{d_1})$ for some $t \in \mathcal{E}$. Observe that by assumption $t \neq e_2$. Thus, we are done after applying φ_1 unless $t = b_3$, so we assume this in the second step.

$$
\begin{array}{ccc}
\boxed{t} & \boxed{c_2} & \boxed{d_1} \\
\boxed{t} & \boxed{c_3} & \boxed{d_2} \\
\boxed{b_4} & \boxed{c_4} & \boxed{d_2}
\end{array}
$$
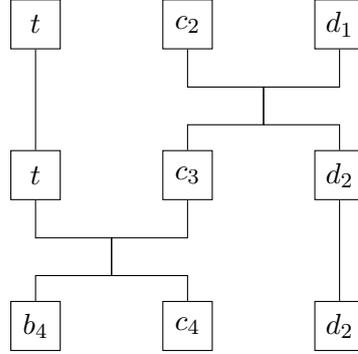
We are done in this case since the pair (c_4, d_2) is stable.

Case 19: We consider the triple $(\mathbf{t}, \mathbf{b_3}, \mathbf{c_3})$ for some $t \in \mathcal{E}$. Observe that by assumption $t \neq a_2$. Thus, we are done after applying φ_1 since all other pairs (t, b_4) are stable.

$$
\begin{array}{ccc}
\boxed{t} & \boxed{b_3} & \boxed{c_3} \\
\boxed{t} & \boxed{b_4} & \boxed{c_4}
\end{array}
$$
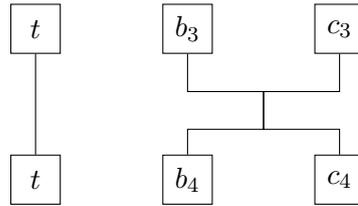
Case 20: We consider the triple $(\mathbf{t}, \mathbf{a_2}, \mathbf{b_4})$ for some $t \in \mathcal{E}$. Here, we are done after φ_1 since all pairs of the form (t, a_1) are stable.

$$
\begin{array}{ccc}
\boxed{t} & \boxed{a_2} & \boxed{b_4} \\
\boxed{t} & \boxed{a_1} & \boxed{b_5}
\end{array}
$$
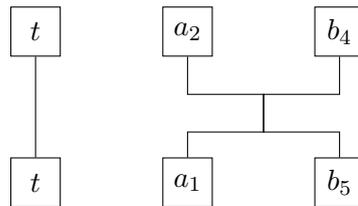
Case 21: We consider the triple $(\mathbf{t}, \mathbf{b_5}, \mathbf{c_4})$ for some $t \in \mathcal{E}$. Here, we are done after applying φ_1 unless $t = a_1$, so we assume this in the second step.

$$\begin{array}{ccc} t & b_5 & c_4 \\ t & b_6 & c_5 \\ 1 & e_3 & c_5 \end{array}$$
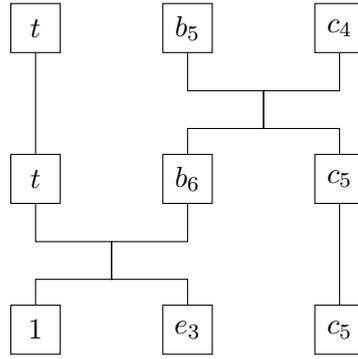
We are done in this case since this generates a 1.

Case 22: We consider the triple $(\mathbf{t}, \mathbf{c_5}, \mathbf{d_2})$ for some $t \in \mathcal{E}$. Observe that by assumption $t \neq e_3$. Thus, we are done after applying φ_1 unless $t = b_6$, so we assume this in the second step.

$$\begin{array}{ccc} t & c_5 & d_2 \\ t & c_6 & d_1 \\ b_1 & c_1 & d_1 \end{array}$$
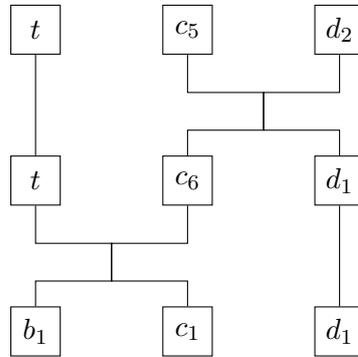
We are done in this case since the pair (c_1, d_1) is stable.

Case 23: We consider the triple $(\mathbf{t}, \mathbf{b_6}, \mathbf{c_6})$ for some $t \in \mathcal{E}$. Observe that by assumption $t \neq a_1$. In this case, we are done after applying φ_1 since all pairs of the form (t, b_1) are stable if $t \neq a_1$.

$$\begin{array}{ccc} t & b_6 & c_6 \\ t & b_1 & c_1 \end{array}$$
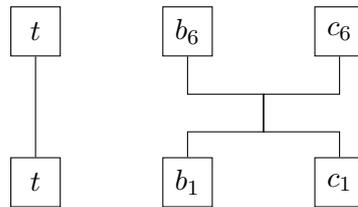
Case 24: We consider the triple $(\mathbf{t}, \mathbf{a_2}, \mathbf{b_3})$ for some $t \in \mathcal{E}$. In this case, we are done after applying φ_1 since it produces a 1.

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$$t \quad\quad a_2 \quad\quad b_3$$
$$t \quad\quad 1 \quad\quad e_2$$
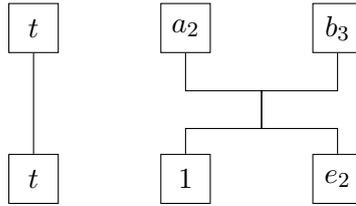
Case 25: We consider the triple $(\mathbf{t}, \mathbf{a_1}, \mathbf{b_6})$ for some $t \in \mathcal{E}$. In this case, we are done after applying φ_1 since it produces a 1.

$$t \quad\quad a_1 \quad\quad b_6$$
$$t \quad\quad 1 \quad\quad e_3$$
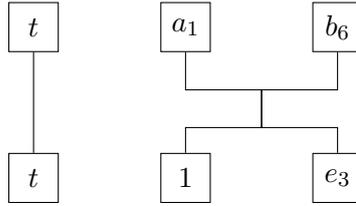
Case 26: We consider the triple $(\mathbf{t}, \mathbf{e_2}, \mathbf{c_2})$ for some $t \in \mathcal{E}$. In this case, we are done after applying φ_1 since all pairs of the form (t, f_2) are stable.

$$t \quad\quad e_2 \quad\quad c_2$$
$$t \quad\quad f_2 \quad\quad g_2$$
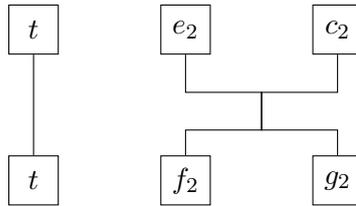
Case 27: We consider the triple $(\mathbf{t}, \mathbf{e_3}, \mathbf{c_5})$ for some $t \in \mathcal{E}$. In this case, we are done after applying φ_1 since all pairs of the form (t, f_3) are stable.

$$t \quad\quad e_3 \quad\quad c_5$$
$$t \quad\quad f_3 \quad\quad g_3$$
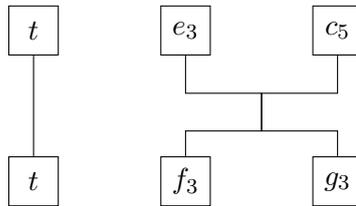
Case 28: We consider the triple $(\mathbf{t}, \mathbf{e_2}, \mathbf{c_3})$ for some $t \in \mathcal{E}$. In this case, we are done after applying φ_1 since all pairs of the form (t, f_3) are stable.

$$t \quad\quad e_2 \quad\quad c_3$$
$$t \quad\quad f_3 \quad\quad g_3$$

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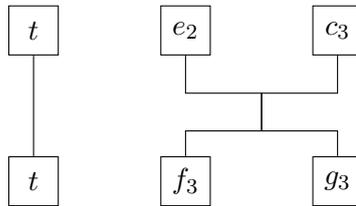
Case 29: We consider the triple $(\mathbf{t}, \mathbf{e_3}, \mathbf{c_6})$ for some $t \in \mathcal{E}$. In this case, we are done after applying φ_1 since all pairs of the form (t, f_2) are stable.

$$
\begin{array}{ccc}
\boxed{t} & \boxed{e_3} & \boxed{c_6} \\
\boxed{t} & \boxed{f_2} & \boxed{g_2}
\end{array}
$$
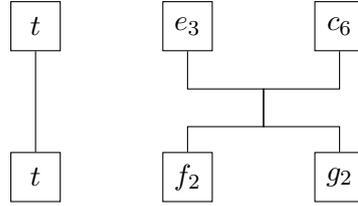
Case 30: We consider the triple $(\mathbf{t}, \mathbf{g_2}, \mathbf{d_1})$ for some $t \in \mathcal{E}$. Here, we are done after applying φ_1 unless $t = f_2$, so we assume this in the second step.

$$
\begin{array}{ccc}
\boxed{t} & \boxed{g_2} & \boxed{d_1} \\
\boxed{t} & \boxed{h_2} & \boxed{i} \\
\boxed{j} & \boxed{k} & \boxed{i}
\end{array}
$$
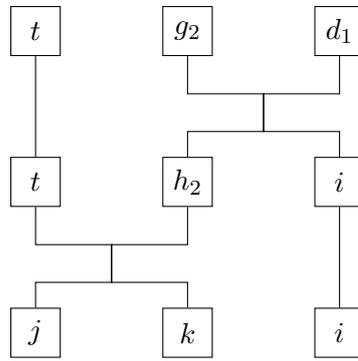
We are done in this case since the pair (k, i) is stable.

Case 31: We consider the triple $(\mathbf{t}, \mathbf{g_3}, \mathbf{d_2})$ for some $t \in \mathcal{E}$. Here, we are done after applying φ_1 unless $t = f_3$, so we assume this in the second step.

$$
\begin{array}{ccc}
\boxed{t} & \boxed{g_3} & \boxed{d_2} \\
\boxed{t} & \boxed{h_3} & \boxed{i} \\
\boxed{j} & \boxed{k} & \boxed{i}
\end{array}
$$
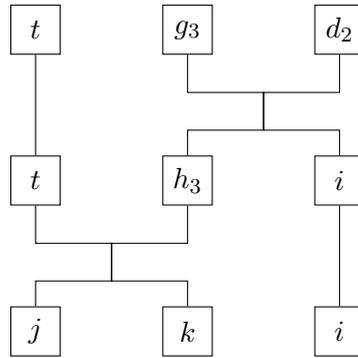
We are done in this case since the pair (k, i) is stable.

Case 32: Last, we consider the triple $(\mathbf{t}, \mathbf{f_n}, \mathbf{h_n})$ for some $t \in \mathcal{E}$ and $n \in \{2, 3\}$. In this case, we are done after applying φ_1 since all pairs of the form (t, j) are stable.
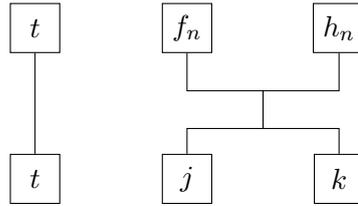
Since the above list contains all triples which are not everywhere stable or contain 1, this completes the proof of the fourth condition.

Now we are going to check the fifth condition for local factorability, the normal form condition. Recall that we have to show that the normal form of a triple remains unchanged under applying φ_1. We will use Lemma 2.3.11 which says we only have to check the totally unstable triples. For the given map φ, these are only the triples (a_2, b_3, c_3), (a_1, b_6, c_6), (e_2, c_2, d_1) and (e_3, c_5, d_2). We use Lemma 2.3.5 to compute normal forms for triples.

In the first case, the application of $\varphi_1\varphi_2\varphi_1\varphi_2$ gives the following picture.
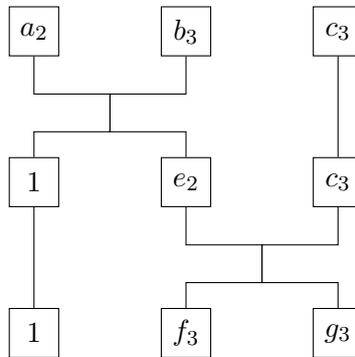
which is everywhere stable and thus already the (extended) normal form.

On the other hand, the application of $\varphi_1\varphi_2\varphi_1\varphi_2\varphi_1$ yields the following.
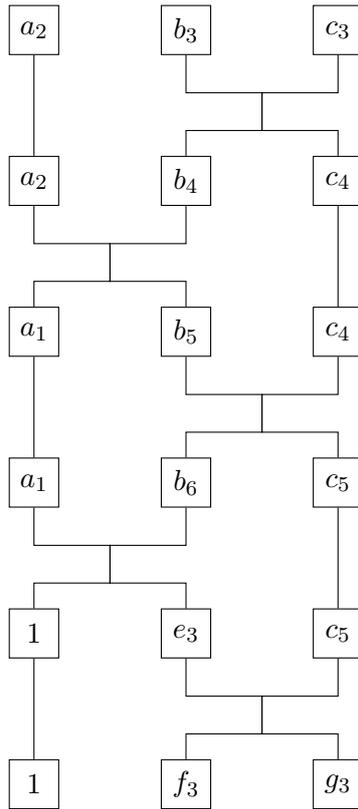
So both normal forms coincide in this first case.

The second case is very similar: The application of $\varphi_1\varphi_2\varphi_1\varphi_2$ gives the following picture.

which is everywhere stable and thus already the (extended) normal form.

On the other hand, the application of $\varphi_1\varphi_2\varphi_1\varphi_2\varphi_1$ yields the following.

a_1	b_6	c_6
a_1	b_1	c_1
a_2	b_2	c_1
a_2	b_3	c_2
1	e_2	c_2
1	f_2	g_2
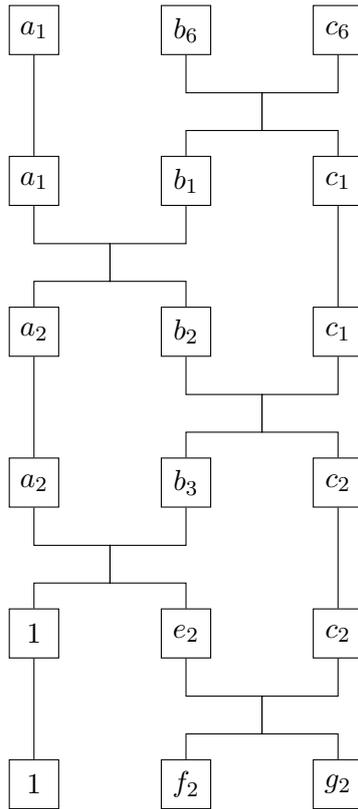
So both normal forms coincide in this case.

We continue with the third case: The normal form of (e_2, c_2, d_1) is obtained as follows.
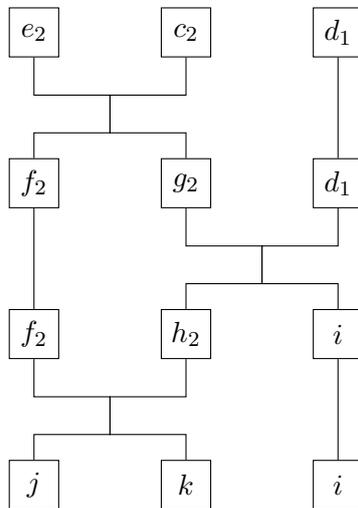
e_2	c_2	d_1
f_2	g_2	d_1
f_2	h_2	i
j	k	i

and the tuple (j, k, i) is already everywhere stable. On the other hand, the application of $\varphi_1\varphi_2\varphi_1\varphi_2\varphi_1$ yields the following.

$$
\begin{array}{ccc}
\boxed{e_2} & \boxed{c_2} & \boxed{d_1} \\
\boxed{e_2} & \boxed{c_3} & \boxed{d_2} \\
\boxed{f_3} & \boxed{g_3} & \boxed{d_2} \\
\boxed{f_3} & \boxed{h_3} & \boxed{i} \\
\boxed{j} & \boxed{k} & \boxed{i}
\end{array}
$$
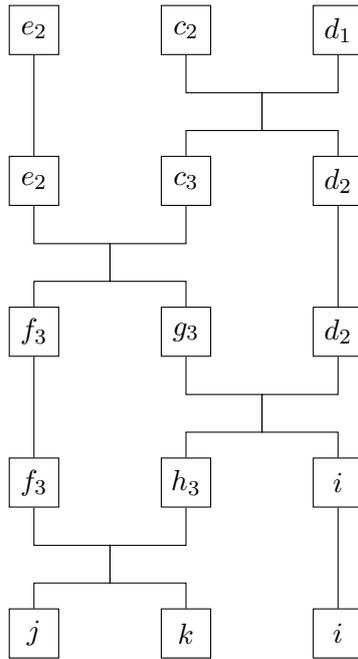
So we obtain in both cases the same normal form. The last case is again very similar: The normal form of (e_3, c_5, d_2) is obtained as follows.
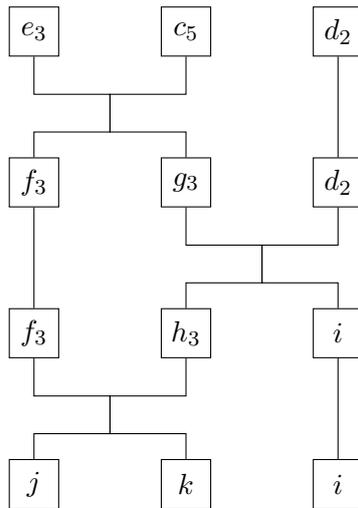
$$
\begin{array}{ccc}
\boxed{e_3} & \boxed{c_5} & \boxed{d_2} \\
\boxed{f_3} & \boxed{g_3} & \boxed{d_2} \\
\boxed{f_3} & \boxed{h_3} & \boxed{i} \\
\boxed{j} & \boxed{k} & \boxed{i}
\end{array}
$$

and the tuple (j, k, i) is already everywhere stable. On the other hand, the application of $\varphi_1 \varphi_2 \varphi_1 \varphi_2 \varphi_1$ yields the following.
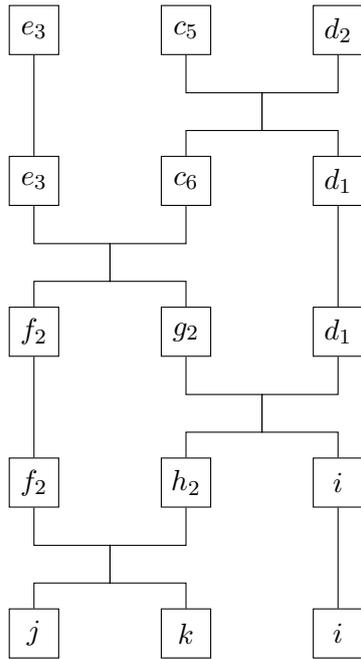
Thus, also in this case, the normal form condition is satisfied. This implies that the map φ defined in this Lemma is indeed a local factorability structure.

Next, observe that the associated rewriting system is not noetherian. Indeed, we have the chain of rewritings

a_1 b_1 c_1 d_1

a_2 b_2 c_1 d_1

a_2 b_3 c_2 d_1

a_2 b_3 c_3 d_2

a_2 b_4 c_4 d_2

a_1 b_5 c_4 d_2

a_1 b_6 c_5 d_2

a_1 b_6 c_6 d_1

a_1 b_1 c_1 d_1

$\ldots$ $\ldots$ $\ldots$ $\ldots$
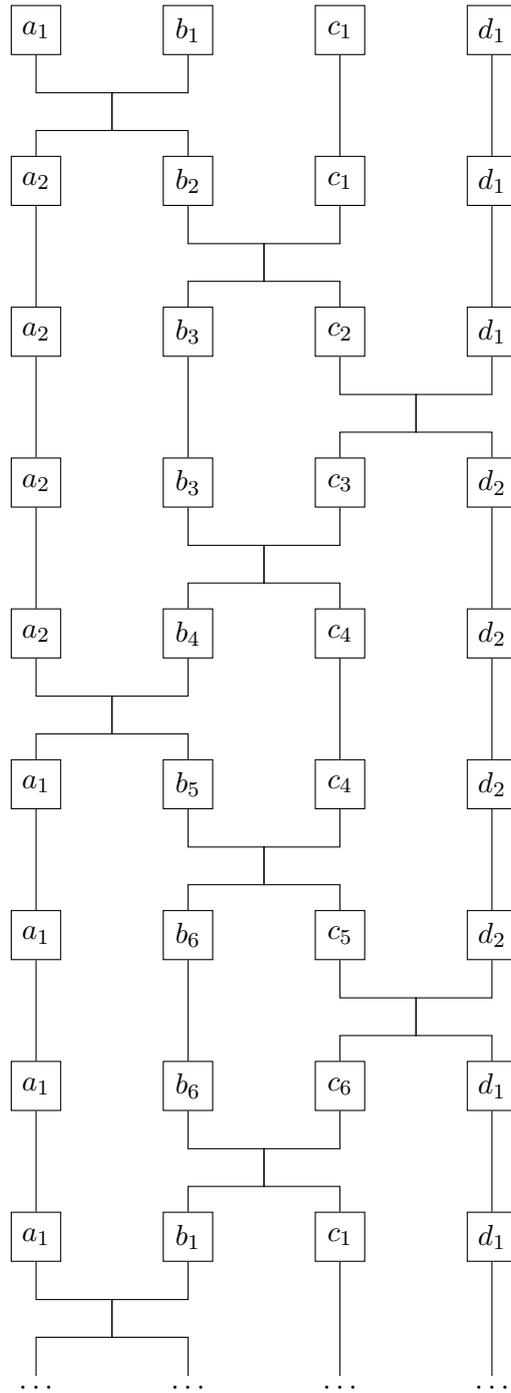
So we obtain a cycle in the rewritings. Observe that this cycle is non-trivial: For example, note that if $a_1 = a_2$, there had to be a zigzag of rewritings from a_1 to a_2. But none of the non-trivial rewriting rules starts or ends with a_1, so this is impossible.

Last, we are going to show that the monoid M defined by the local factorability

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structure as above is right-cancellative. We are going to rely on the φ-normal forms in this monoid. Assume M is not right-cancellative. Then there are some elements $x, y, z \in M$ such that $xz = yz$, but $x \neq y$. Obviously, $z \neq 1$. We want to consider an example with minimal $\mathcal{E}$-word length of z. Then $N_{\mathcal{E}}(z) = 1$: Indeed, otherwise there is an $s \in \mathcal{E}$ and $w \in M \setminus \{1\}$ such that $z = ws$. Then $(xw)s = (yw)s$, so that either $xw = yw$ contradicting the minimality of z, or $xw \neq yw$, which is again contradicting the minimality.

So we know that there are $x \neq y \in M$ and $z \in \mathcal{E}$ such that $xz = yz$. Let $(x_m, \ldots, x_1)$ be the normal form of x and $(y_n, \ldots, y_1)$ the normal form of y. We may choose an example where $m + n$ is minimal. First, we want to demonstrate that $m + n \geq 3$. Indeed, $m + n$ has to be at least 1 by definition. Then, if $m = 1$ and $n = 0$ (the other case can be treated symmetrically), we have $x_1 z = z$, which in particular implies $\varphi(x_1, z) = (1, z)$. But there is no pair (x_1, z) with $x_1 \neq 1$ and image $(1, z)$ under φ, so this cannot happen. More generally, we can exclude the case $n = 0$ (and so symmetrically $m = 0$): We compute the normal form of $(x_m, \ldots, x_1, z)$ using the definition. We already know that this normal form has to be z. Note that in the step where the number of non-trivial letters in the string reduces to one, we have to get a 1 out of a pair of elements by applying φ, so that z must be e_i with $i \in \{2, 3\}$. But all pairs of the form (x_1, e_i) are stable, so that $(x_m, \ldots, x_1, e_i)$ has to be already the normal form, contradicting the assumption $m + n > 0$. Furthermore, we can exclude the case $m = n = 1$, observing that in the definition of φ, there are no distinct pairs $(x_1, z), (y_1, z)$ which are mapped to the same pair by φ.

Now we know $m + n \geq 3$. In order to compute the normal forms of tuples $(x_m, \ldots, x_1, z)$ and $(y_n, \ldots, y_1, z)$, we first have to apply $\varphi_m \varphi_{m-1} \ldots \varphi_1$ or $\varphi_n \varphi_{n-1} \ldots \varphi_1$, respectively. Observe that if both (x_1, z) and (y_1, z) are stable pairs, then we would have two different normal forms for the same element $xz = yz$ of M, yielding a contradiction. So we may assume $\varphi(x_1, z) = (u_1, v_1)$ with $(x_1, z) \neq (u_1, v_1)$. Note that this in particular implies $z \neq v_1$ by the definition of φ.

Consider now the case where applications of both $\varphi_m \varphi_{m-1} \ldots \varphi_1$ and $\varphi_n \varphi_{n-1} \ldots \varphi_1$ do not produce a 1. Then the results have to be equal, in particular, $n = m$. Furthermore, this implies $\varphi(x_1, z) = (u_1, v_1)$ and $\varphi(y_1, z) = (t_1, v_1)$ with the same v_1, in particular, $(y_1, z) \neq (t_1, v_1)$. There are no two distinct pairs with same right letters in the list defining φ, i.e., if $\varphi(\alpha, \beta) = (\gamma, \delta)$ and $\varphi(\widetilde{\alpha}, \beta) = (\widetilde{\gamma}, \delta)$ and $\beta \neq \delta$, we know that $\widetilde{\alpha} = \alpha$ and $\widetilde{\gamma} = \gamma$. Hence, we may conclude that $x_1 = y_1$ and $u_1 = t_1$. Now since $x \neq y$, we know that $x_m \ldots x_2 \neq y_n \ldots y_2$. The normal forms of $(x_m, \ldots, x_2, u_1)$ and $(y_n, \ldots, y_2, u_1)$ have to coincide since they are the same as the normal form of $xz = yz$ with the right-most letter v_1 deleted, and so the elements $x_m \ldots x_2 u_1$ and $y_n \ldots y_2 u_1$ have to coincide. But this contradicts the minimality assumption on $n + m$.

So we have to consider the case where we obtain a 1 while building the normal form. We may assume that application of φ_p to $\varphi_{p-1} \ldots \varphi_1(x_n, x_{n-1}, \ldots, x_1, z)$ produces the first 1. Define u_q and v_q inductively via $u_0 := z$ and $\varphi(x_{q+1}, u_q) = (u_{q+1}, v_{q+1})$ for $0 \leq q \leq p - 1$. By assumption, $u_p = 1$. (We illustrate the situation by the following picture.)

x_n	$\cdots$	x_p	x_{p-1}	$\cdots$	x_3	x_2	x_1	z
							φ_1	
x_n	$\cdots$	x_p	x_{p-1}	$\cdots$	x_3	x_2	u_1	v_1
						φ_2		
x_n	$\cdots$	x_p	x_{p-1}	$\cdots$	x_3	u_2	v_2	v_1
					φ_3			
x_n	$\cdots$	x_p	x_{p-1}	$\cdots$	u_3	v_3	v_2	v_1
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
x_n	$\cdots$	x_p	u_{p-1}	$\cdots$	v_4	v_3	v_2	v_1
		φ_p						
x_n	$\cdots$	1	v_p	$\cdots$	v_4	v_3	v_2	v_1
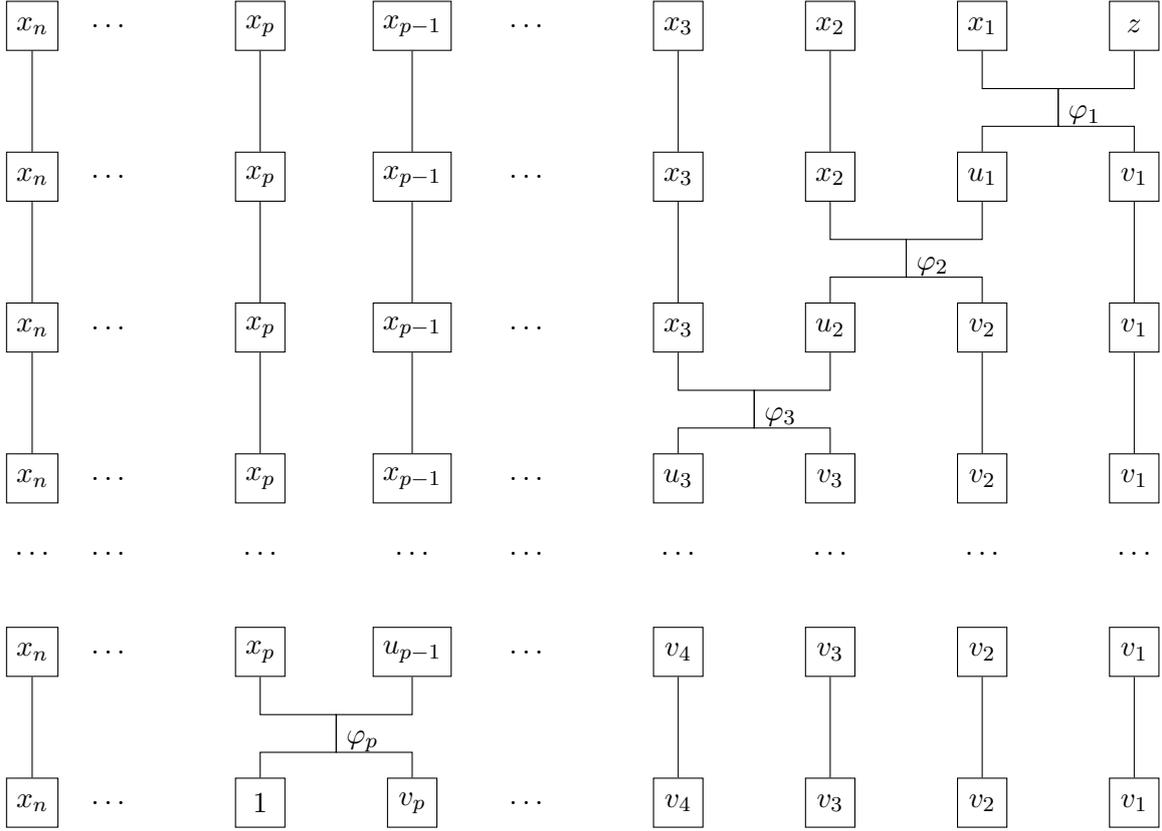
Observe that by definition of the normal form, we have

$$\mathrm{NF}(x_{p-1}, \ldots x_1, z) = \varphi_{p-1} \ldots \varphi_1(x_{p-1}, \ldots x_1, z) = (u_{p-1}, v_{p-1}, \ldots, v_1).$$

In particular, this tuple is totally stable.

Now since $\varphi(x_p, u_{p-1}) = (1, v_p)$ and we know that $x_p \neq 1$ and $u_{p-1} \neq 1$, we conclude that $v_p = e_r$ with $r \in \{2, 3\}$. Thus, the pair (x_p, u_{p-1}) equals (a_{4-r}, b_{3r-3}). Furthermore, observe that there is no unstable pair which ends with e_r on the right, so the tuple $(x_m, \ldots, x_{p+1}, e_r)$ is totally stable.

If we assume $p = 1$, then $z = b_{3r-3}$ and $x_1 = a_{4-r}$ and $(x_m, \ldots, x_2, e_r)$ is the normal form of $xz = yz$. So in particular, this is the normal form of the tuple $(y_n, \ldots, y_1, b_{3r-3})$. In particular, since $b_{3r-3} \neq e_r$, the tuple (y_1, b_{3r-3}) has to be unstable, so that $y_1 = a_{4-r}$ follows. This now implies that $y_n \ldots y_2 e_r = x_m \ldots x_2 e_r$, while $y_n \ldots y_2 \neq x_m \ldots x_2$, contradicting the minimality of $n + m$.

So we have shown that $p \geq 2$. To simplify the notation, we consider first the case $r = 2$. Here, we know that $x_p = a_2$ and $u_{p-1} = b_3$. Since the pair (x_p, x_{p-1}) was stable, we know that x_{p-1} is nor b_3 neither b_4. In particular, the pair (x_{p-1}, u_{p-2}) cannot be stable since $u_{p-1} \neq x_{p-1}$. Thus, (x_{p-1}, u_{p-2}) has to be an unstable pair such that $\varphi(x_{p-1}, u_{p-2}) = (b_3, v_{p-1})$. This implies $x_{p-1} = b_2$, $u_{p-2} = c_1$ and $v_{p-1} = c_2$.

If $p = 2$, then $z = c_1$ and we have

$$
\begin{aligned}
\varphi_m \varphi_{m-1} \ldots \varphi_1(x_m, \ldots, x_1, z) &= \varphi_m \varphi_{m-1} \ldots \varphi_1(x_m, \ldots, x_3, a_2, b_2, c_1) \\
&= \varphi_m \varphi_{m-1} \ldots \varphi_2(x_m, \ldots, x_3, a_2, b_3, c_2) \\
&= \varphi_m \varphi_{m-1} \ldots \varphi_3(x_m, \ldots, x_3, 1, e_2, c_2) \\
&= (1, x_m, \ldots, x_3, e_2, c_2).
\end{aligned}
$$

Computing the normal form of this tuple is done after applying φ_1 since then we obtain the tuple $(x_m, \ldots, x_3, f_2, g_2)$ and all pairs of the form (x_3, f_2) are stable. So we know that $(y_n, \ldots, y_1, c_1)$ is not in the normal form. In particular, the pair (y_1, c_1) is unstable and so $y_1 = b_2$, so that $\varphi_n \ldots \varphi_1$ yields a c_2 in the right-most place when applied to $(y_n, \ldots, y_1, c_1)$. Since the normal form of this tuple ends with g_2 on the right, we know that applying $\varphi_n \ldots \varphi_1$ must have produced a 1. If y_2 would not be equal to a_2, the triple $(y_2, \varphi(y_1, c_1)) = (y_2, b_3, c_2)$ would be stable and thus the whole tuple $(y_n, \ldots, y_2, b_3, c_2)$, which contradicts our assumptions. So $y_2 = a_2$, and thus we have the equality

$$
\begin{aligned}
(y_n \ldots y_3 e_2) c_2 &= (y_n \ldots y_3 a_2 b_3) c_2 = (y_n \ldots y_3 a_2) b_3 c_2 = y_n \ldots y_3 y_2 b_2 c_1 \\
&= y_n \ldots y_3 y_2 y_1 z = x_m \ldots x_1 z = (x_m \ldots x_3 e_2) c_2.
\end{aligned}
$$

Now since $x_1 = y_1$ and $x_2 = y_2$ and $x \neq y$, we conclude that $x_m \ldots x_3 \neq y_n \ldots y_3$. Then either $(y_n \ldots y_3) e_2 = (x_m \ldots x_3) e_2$ and this produces a counterexample to right-cancellativity contradicting the minimality of $n + m$, or $(y_n \ldots y_3) e_2 \neq (x_m \ldots x_3) e_2$, then we have a contradictory counterexample due to

$$
(y_n \ldots y_3 e_2) c_2 = (x_m \ldots x_3 e_2) c_2.
$$

So we know that $p \geq 3$. Recall that $x_p = a_2$, $u_{p-1} = b_3$, $x_{p-1} = b_2$, $u_{p-2} = c_1$ and $v_{p-1} = c_2$. Now the pair (x_{p-1}, x_{p-2}) is stable, thus x_{p-2} is not c_1. As before, this implies that the pair (x_{p-2}, u_{p-3}) is unstable and is mapped via φ to (c_1, v_{p-2}). Yet, this is impossible! This completes the proof for the case $r = 2$.

The proof in case $r = 3$ is completely analogous. This can be also seen as follows: There is a map $\gamma \colon M \to M$, defined below by its values on generators, which is a monoid homomorphism and involution and which maps e_2 to e_3 and preserves φ. This allows to

avoid the symmetrical argument. The map γ is given as follows.

$$
\begin{aligned}
a_1 &\mapsto a_2 \\
a_2 &\mapsto a_1 \\
b_1 &\mapsto b_4 \\
b_2 &\mapsto b_5 \\
b_3 &\mapsto b_6 \\
b_4 &\mapsto b_1 \\
b_5 &\mapsto b_2 \\
b_6 &\mapsto b_3 \\
c_1 &\mapsto c_4 \\
c_2 &\mapsto c_5 \\
c_3 &\mapsto c_6 \\
c_4 &\mapsto c_1 \\
c_5 &\mapsto c_2 \\
c_6 &\mapsto c_3 \\
d_1 &\mapsto d_2 \\
d_2 &\mapsto d_1 \\
e_2 &\mapsto e_3 \\
e_3 &\mapsto e_2 \\
f_2 &\mapsto f_3 \\
f_3 &\mapsto f_2 \\
g_2 &\mapsto g_3 \\
g_3 &\mapsto g_2 \\
h_2 &\mapsto h_3 \\
h_3 &\mapsto h_2 \\
i &\mapsto i \\
j &\mapsto j \\
k &\mapsto k
\end{aligned}
$$

This completes the proof.

$\square$

7.2 Remark on Stable Pairs

When we consider a local factorability structure associated to a general factorability structure, we may ask whether the normal form of some tuple $(x_n, \ldots, x_1)$ is related to the normal form of the subtuple $(x_{n-1}, \ldots, x_1)$. The aim of this section is to provide an example where the normal form of the subtuple is not a subtuple of the normal form.

Indeed, we show that this can even happen for triples and even under the assumption that the right-most letter is already the right-most letter of the normal form.

Proposition 7.2.1. *Consider the monoid*

$$M = \langle a, b, c, d, e, f, g, h, i, j \mid ab = de, af = hi, bc = fg, hj = de, ig = jc \rangle.$$

Set $\mathcal{E} = \{a, b, c, d, e, f, g, h, i, j\}$ and $\mathcal{E}^+ = \mathcal{E} \cup \{1\}$. Consider the map

$$\varphi \colon \mathcal{E}^+ \times \mathcal{E}^+ \to \mathcal{E}^+ \times \mathcal{E}^+$$

assigning the left-hand side of each relation its right-hand side, with $\varphi(x, 1) = (1, x)$ for all $x \in \mathcal{E}^+$ and identity otherwise. Then φ defines a local factorability structure on M with respect to the generating system $\mathcal{E}$.

Proof. We use the criterion of Corollary 2.3.12. Observe that by definition, the first three conditions are satisfied. We are going to show

$$\varphi_2 \varphi_1 \varphi_2 \varphi_1 = \varphi_2 \varphi_1 \varphi_2 = \varphi_1 \varphi_2 \varphi_1 \varphi_2 \colon \mathcal{E}^3 \to \mathcal{E}^3.$$

Apply the maps to a triple $(x, y, z) \in \mathcal{E}^3$. Observe that if a triple is stable, any combination of φ_1 and φ_2 leaves it unchanged. Thus, all three maps are identities if both pairs are stable. We have to consider the other cases. The only unstable pairs of length 2 are $(a, b), (a, f), (b, c), (h, j), (i, g)$. Note furthermore that the first equality holds automatically if the pair (y, z) is stable.

Consider now the possible cases for $(\mathbf{x}, \mathbf{y}, \mathbf{z})$.

Case 1: For $(\mathbf{a}, \mathbf{b}, \mathbf{z})$, an application of φ_2 turns it into (d, e, z). This triple is stable since no unstable pair starts with e.

The first equality holds except for possibly $z = c$ since in other cases, (b, z) is stable. In this case, we have

$$
\begin{aligned}
\varphi_2 \varphi_1 \varphi_2 \varphi_1(a, b, c) &= \varphi_2 \varphi_1 \varphi_2(a, f, g) \\
&= \varphi_2 \varphi_1(h, i, g) = \varphi_2(h, j, c) = (d, e, c)
\end{aligned}
$$

and, on the other hand, $\varphi_2(a, b, c) = (d, e, c)$. The triple (d, e, c) is everywhere stable so that the first equality holds for (a, b, c).

Case 2: For $(\mathbf{a}, \mathbf{f}, \mathbf{z})$, an application of φ_2 turns it into (h, i, z). This triple is stable unless $z = g$; in this case, we have $\varphi_1(h, i, g) = (h, j, c)$ and $\varphi_2 \varphi_1 \varphi_2(a, f, g) = (d, e, c)$ is everywhere stable so that the second equality holds.

The first equality is true for any z since there are no unstable pairs starting with f.

Case 3: For $(\mathbf{b}, \mathbf{c}, \mathbf{z})$, an application of φ_2 turns it into (f, g, z). This triple is stable since no unstable triple starts with a g.

The first equality is true for any z since there are no unstable pairs starting with c.

Case 4: For $(\mathbf{h}, \mathbf{j}, \mathbf{z})$, an application of φ_2 yields (d, e, z) which is stable since no unstable pair starts with e.

The first equality is also true for any z since all pairs starting with j are stable.

Case 5: For $(\mathbf{i}, \mathbf{g}, \mathbf{z})$, an application of φ_2 yields (j, c, z), which is stable since no unstable pair starts with c.

For the first equality, observe that also all pairs starting with g are stable.

From now on, we may assume that the left pair (x, y) is stable; in particular, applying φ_2 immediately doesn't change the triple.

Case 6: For $(\mathbf{x}, \mathbf{a}, \mathbf{b})$, an application of φ_1 yields the triple (x, d, e), which is stable since no unstable pair ends with d.

Case 7: For $(\mathbf{x}, \mathbf{a}, \mathbf{f})$, an application of φ_1 gives us the triple (x, h, i), which is stable since no unstable pair ends with h.

Case 8: For $(\mathbf{x}, \mathbf{b}, \mathbf{c})$, an application of φ_1 ends with (x, f, g). This triple is stable unless $x = a$, but this case was already excluded since (a, b) is unstable (we have already considered the triple (a, b, c) in Case 1).

Case 9: For $(\mathbf{x}, \mathbf{h}, \mathbf{j})$, an application of φ_1 gives (x, d, e), which is stable since no unstable pair ends with d.

Case 10: For $(\mathbf{x}, \mathbf{i}, \mathbf{g})$, we apply φ_1 and obtain (x, j, c). This triple is stable unless $x = h$. In this case, $\varphi_2 \varphi_1(h, i, g) = \varphi_2(h, j, c) = (d, e, c)$ is stable. The three compositions coincide in this case.

$\square$

Observe that in M, the normal form $\mathrm{NF}(a, b, c)$ equals (d, e, c) but $\mathrm{NF}(b, c) = (f, g)$. In particular, the normal form of some tuple does not have the normal form of its subtuple on the right.

7.3 Finiteness of Q'_n

The content of this section is joint work with A. Heß.

Our aim is to prove the following theorem. It is closely connected to the proof that the rewriting system defined by a factorability structure is noetherian in some cases. We will use the notation of Section 2.2.

Theorem 7.3.1. *The monoids Q'_n are finite for every n.*

The proof will proceed in several steps. We will first show the following important proposition:

Proposition 7.3.2. *For all n, the statements $A(n)$, $B(n)$, $C(n)$ hold.*

A(n) *Let I be a sequence in F_n with $I = 1\widetilde{X}n$ and $\widetilde{X} = \mathrm{sh}_1(X)$ for some $X \in F_{n-2}$ (in other words, 1 and n do not occur in $\widetilde{X}$). Furthermore, we assume $I_1^n \subset I$. Then there are sequences $J, K \in F_{n-2}$ such that $I \sim \mathrm{sh}_2(J)I_1^n \mathrm{sh}_1(K)$. This can be rephrased as follows: In Q'_n, if a representing sequence I starts with 1, ends with n, contains all other letters in increasing order in between (not necessarily as a connected subsequence), and no 1's and n's occur in between, then I is equivalent to another representative which contains I_1^n as a connected subsequence, only entries greater than 2 left to I_1^n and entries between 2 and $n-1$ on the right of it.*

B(n) *Let Z be a sequence in F_{n-1} and set $\widetilde{Z} = \mathrm{sh}_1 Z$. Then we have $1\widetilde{Z}I_1^n \sim \widetilde{Z}I_1^n$.*

C(n) *Let L be any sequence in F_n and let M be a sequence obtained from L by deleting all 1's in L. Then $LI_1^n \sim MI_1^n$.*

Proof. The statement $C(n)$ is an easy consequence of $B(n)$ for any fixed n. Indeed, we can write L as

$$L = \mathrm{sh}_1(L_m)1\,\mathrm{sh}_1(L_{m-1})1\ldots\mathrm{sh}_1(L_1)1\,\mathrm{sh}_1(L_0)$$

with $L_i \in F_{n-1}$. Now we can iteratively apply $B(n)$ with $Z = L_i L_{i-1} \ldots L_1 L_0$ to delete the 1's.

For $n = 1$ and $n = 2$, X in $A(1)$ has to be the empty sequence, so $I = 1 = I_1^n$ and $A(1)$ holds. Similarly, for $n = 1$, the string Z in $B(1)$ is empty, so $B(1)$ holds. For $n = 3$, the only non-empty case of $A(3)$ is $X = 1$. Then $I = (1\,2\,3)$, and the statement $A(3)$ obviously holds.

We now prove $A(n)$ and $B(n)$ by induction. Assume $A(k)$ and $B(k)$ hold for all $1 \le k \le n-1$. We prove first $B(n)$. So let Z be in F_{n-1} and set $\widetilde{Z} = \mathrm{sh}_1 Z$. We would like to show $1\widetilde{Z}I_1^n \sim \widetilde{Z}I_1^n$.

Without loss of generality we may assume that Z is left-most due to Proposition 2.2.8. If Z does not contain a 1, all entries of $\widetilde{Z}$ are at least 3 and $1\widetilde{Z} \sim_P \widetilde{Z}1$, so we conclude

$$1\widetilde{Z}I_1^n \sim_P \widetilde{Z}1I_1^n \sim_P \widetilde{Z}I_1^n$$

as I_1^n starts with a 1. Now assume that Z does contain a 1. Then we can write $Z = Z'Z''$ with $Z' \in F_{n-1}$ not containing a 1 and $Z'' \in F_{n-1}$ starting with a 1. If $\widetilde{Z'} = \mathrm{sh}_1(Z')$ and $\widetilde{Z''} = \mathrm{sh}_1(Z'')$, then $\widetilde{Z'}$ again contains only entries commuting with 1, so it is enough to show that $1\widetilde{Z''}I_1^n \sim \widetilde{Z''}I_1^n$ since then we have

$$1\widetilde{Z}I_1^n = 1\widetilde{Z'}\widetilde{Z''}I_1^n \sim_P \widetilde{Z'}1\widetilde{Z''}I_1^n \sim \widetilde{Z'}\widetilde{Z''}I_1^n = \widetilde{Z}I_1^n.$$

Let r be the maximal entry occurring in Z''; by assumptions $r \le n-1$. By the definition of r, we know that $Z'' \in F_r$. There must be a connected subsequence of Z'' starting with 1, ending with r and not containing 1 or r in between: Find the first occurrence of r in Z'' starting from the left, then there is at least one 1 left from it since Z'' starts with 1; from all the 1's left to this r take the one on the very right. So we may write Z'' as $Z'' = U1\widetilde{V}rW$ with $U, W \in F_r$, $\widetilde{V} = \mathrm{sh}_1(V)$ with $V \in F_{r-2}$. Since $1\widetilde{V}r$ is a connected

subsequence of Z'' and Z'' is a connected subsequence of the left-most sequence Z, we conclude that $1\widetilde{V}r$ is left-most. This implies $I_1^r \subset 1\widetilde{V}r$ by Lemma 2.2.7. Then we may apply $A(r)$ and get two sequences $J, K \in F_{r-2}$ such that $1\widetilde{V}r \sim \mathrm{sh}_2(J)I_1^r \mathrm{sh}_1(K)$. Therefore, $Z'' \sim U\mathrm{sh}_2(J)I_1^r \mathrm{sh}_1(K)W$.

Observe that all entries of $\mathrm{sh}_2(J)$ are at least 3. By $C(r)$, which holds since we assumed $B(r)$, we may also change U to $\widetilde{U'}$ by deleting all 1's in U so that $\widetilde{U'} = \mathrm{sh}_1(U')$ for some $U' \in F_{r-1}$.

Now we put together what we have done so far:

$$1\,\mathrm{sh}_1(Z'')I_1^n \sim 1\,\mathrm{sh}_2(U')\,\mathrm{sh}_3(J)I_2^{r+1}\,\mathrm{sh}_2(K)\,\mathrm{sh}_1(W)I_1^{r+1}I_{r+2}^n.$$

Consider now the product of $\mathrm{sh}_2(D_{r-1})$ with the left-hand side. It is obvious that $\mathrm{sh}_2(D_{r-1})$ commutes with 1 by $\sim_P$ and the 1 does not occur in $\mathrm{sh}_2(D_{r-1})$. It is enough to show that

$$\mathrm{sh}_2(D_{r-1})1\,\mathrm{sh}_2(U')\,\mathrm{sh}_3(J)I_2^{r+1}\,\mathrm{sh}_2(K)\,\mathrm{sh}_1(W)I_1^{r+1}I_{r+2}^n \sim_Q$$
$$\mathrm{sh}_2(D_{r-1})\,\mathrm{sh}_2(U')\,\mathrm{sh}_3(J)I_2^{r+1}\,\mathrm{sh}_2(K)\,\mathrm{sh}_1(W)I_1^{r+1}I_{r+2}^n$$

since that by definition of $\sim$ in Definition 2.2.1 implies

$$1\,\mathrm{sh}_2(U')\,\mathrm{sh}_3(J)I_2^{r+1}\,\mathrm{sh}_2(K)\,\mathrm{sh}_1(W)I_1^{r+1}I_{r+2}^n \sim$$
$$\mathrm{sh}_2(U')\,\mathrm{sh}_3(J)I_2^{r+1}\,\mathrm{sh}_2(K)\,\mathrm{sh}_1(W)I_1^{r+1}I_{r+2}^n.$$

Now we have

$$\mathrm{sh}_2(D_{r-1})1\,\mathrm{sh}_2(U')\,\mathrm{sh}_3(J)I_2^{r+1}\,\mathrm{sh}_2(K)\,\mathrm{sh}_1(W)I_1^{r+1}I_{r+2}^n \sim_P$$
$$1\,\mathrm{sh}_2(D_{r-1}U'\,\mathrm{sh}_1(J))I_2^{r+1}\,\mathrm{sh}_2(K)\,\mathrm{sh}_1(W)I_1^{r+1}I_{r+2}^n \sim_Q$$
$$1\,\mathrm{sh}_2(D_{r-1})I_2^{r+1}\,\mathrm{sh}_2(K)\,\mathrm{sh}_1(W)I_1^{r+1}I_{r+2}^n,$$

where the first equivalence follows from the fact that $\mathrm{sh}_2(D_{r-1})$ has entries greater or equal to 3, and the second one from the absorption property 2.2.4 of D_{r-1} in Q_{r-1}, where U' and $\mathrm{sh}_1(J)$ define elements in.

Observe now that we can write $\mathrm{sh}_2(D_{r-1})I_2^{r+1}$ as $\mathrm{sh}_1(D_r)$. Again by Theorem 2.2.4, we have

$$\mathrm{sh}_1(D_r\,\mathrm{sh}_1(K)W) \sim_Q \mathrm{sh}_1(D_r)$$

since $\mathrm{sh}_1(K), W \in F_r$ represent elements in Q_r. Together this yields

$$1\,\mathrm{sh}_2(D_{r-1})I_2^{r+1}\,\mathrm{sh}_2(K)\,\mathrm{sh}_1(W)I_1^{r+1}I_{r+2}^n \quad \sim_Q \quad 1\,\mathrm{sh}_1(D_r)I_1^{r+1}I_{r+2}^n$$
$$= \quad 1D_{r+1}I_{r+2}^n \sim_Q D_{r+1}I_{r+2}^n,$$

where we again used the absorption property in the last step.

Using the same arguments, one observes that also

$$\mathrm{sh}_2(D_{r-1})\,\mathrm{sh}_2(U')\,\mathrm{sh}_3(J)I_2^{r+1}\,\mathrm{sh}_2(K)\,\mathrm{sh}_1(W)I_1^{r+1}I_{r+2}^n \sim_Q D_{r+1}I_{r+2}^n$$

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implying the claim $B(n)$.

Now we show that also $A(n)$ holds. Suppose we have $X \in F_{n-2}$ such that $I = 1\widetilde{X}n$ contains I_1^n as a (possibly) disconnected subsequence with $\widetilde{X} = \mathrm{sh}_1(X)$. Without loss of generality we may assume X to be left-most; due to Lemma 2.2.9, the property $I_1^{n-2} \subset X$ is preserved. Write X as $X = \widetilde{X}_t 1 \widetilde{X_{t-1}} 1 \ldots 1 \widetilde{X}_2 1 \widetilde{X}_1$ where $\widetilde{X}_i = \mathrm{sh}_1(X_i)$, with $X_i \in F_{n-3}$. The sequences X_t and X_1 are possibly empty, whereas X_i can be assumed non-empty for $2 \leq i \leq t-1$. (Observe that X has to contain a 1 since $I_1^n \subset I = 1\widetilde{X}n$, and that the X_i are left-most again.) Set $\widetilde{\widetilde{X}}_i = \mathrm{sh}_2(X_i)$. Then

$$I = 1\widetilde{\widetilde{X}}_t 2 \widetilde{\widetilde{X_{t-1}}} 2 \ldots 2 \widetilde{\widetilde{X}}_2 2 \widetilde{\widetilde{X}}_1 n.$$

Since $I_1^n \subset I$, there must be a smallest k such that $n-1 \in \widetilde{\widetilde{X}}_k$, i.e. $n-3 \in X_k$. As we already showed $A(n)$ for $n \leq 3$, we may assume $n \geq 4$. Thus, n commutes with 2, and

$$I \sim_P 1\widetilde{\widetilde{X}}_t 2 \widetilde{\widetilde{X_{t-1}}} 2 \ldots 2 \widetilde{\widetilde{X_{k+1}}} 2 \widetilde{\widetilde{X}}_k n 2 \widetilde{\widetilde{X_{k-1}}} 2 \widetilde{\widetilde{X_{k-2}}} 2 \ldots 2 \widetilde{\widetilde{X}}_1$$

Note that $k < t$ since $I_1^n \subset 1\widetilde{X}n$, so there must be an entry $n-1$ after the first appearance of 2. Now we have $2\widetilde{\widetilde{X}}_k n = \mathrm{sh}_1(1\widetilde{X}_k(n-1))$, where $X_k \in F_{(n-1)-2}$. Moreover, $2\widetilde{\widetilde{X}}_k$ is left-most and contains $(2, n-1)$ by assumptions, thus it contains I_2^{n-1} by Lemma 2.2.7 and therefore $I_1^{n-1} \subset 1\widetilde{X}_k(n-1)$. So we may use $A(n-1)$ and obtain sequences $J', K' \in F_{n-3}$ such that $1\widetilde{X}_k(n-1) \sim \mathrm{sh}_2(J')I_1^{n-1}\mathrm{sh}_1(K')$. Putting this into the formula above leads to

$$I \sim 1\widetilde{\widetilde{X}}_t 2 \widetilde{\widetilde{X_{t-1}}} 2 \ldots 2 \widetilde{\widetilde{X_{k+1}}} \mathrm{sh}_3(J') I_2^n \mathrm{sh}_2(K') 2 \widetilde{\widetilde{X_{k-1}}} 2 \widetilde{\widetilde{X_{k-2}}} 2 \ldots 2 \widetilde{\widetilde{X}}_1.$$

Use now $\mathrm{sh}_1(C(n-1))$ to see

$$I \sim 1\widetilde{\widetilde{X}}_t \widetilde{\widetilde{X_{t-1}}} \ldots \widetilde{\widetilde{X_{k+1}}} \mathrm{sh}_3(J') I_2^n \mathrm{sh}_2(K') 2 \widetilde{\widetilde{X_{k-1}}} 2 \widetilde{\widetilde{X_{k-2}}} 2 \ldots 2 \widetilde{\widetilde{X}}_1.$$

Since all entries in $\widetilde{\widetilde{X}}_t \widetilde{\widetilde{X_{t-1}}} \ldots \widetilde{\widetilde{X_{k+1}}} \mathrm{sh}_3(J')$ are at least 3, this term commutes with 1 and we obtain

$$I \sim \mathrm{sh}_2(X_t X_{t-1} \ldots X_{k+1} \mathrm{sh}_1(J')) I_1^n \mathrm{sh}_1(\mathrm{sh}_1(K')1\widetilde{X_{k-1}}1\widetilde{X_{k-2}}1 \ldots 1\widetilde{X}_1).$$

Since $J := X_t X_{t-1} \ldots X_{k+1} \mathrm{sh}_1(J')$ is in F_{n-2} as well as

$$K := \mathrm{sh}_1(K')1\widetilde{X_{k-1}}1\widetilde{X_{k-2}}1 \ldots 1\widetilde{X}_1,$$

this completes the proof of the proposition. $\square$

The following corollary was already announced by A. Heß ([42]).

Corollary 7.3.3. *In Q'_n, the equality $I_1^n I_1^n = I_2^n I_1^n$ follows from $C(n)$ with $L = I_1^n$. Applying the statement $C(n)$ iteratively, we obtain furthermore*

$$(I_1^n)^n \sim I_n^n I_{n-1}^n \ldots I_2^n I_1^n = D_n.$$

Now we are ready to prove that Q'_n is finite.

Proof. (of the Theorem 7.3.1)

We proceed by induction. For $n = 1$, we have $Q'_1 = \{[\varnothing], [1]\}$, so Q'_1 is finite. For $n = 2$, we have

$$Q'_2 = \{[\varnothing], [1], [2], [12], [21], [121], [212]\}.$$

Now assume we have proven the statement for all smaller numbers than n. Set q'_{n-1} to be the number of elements in Q'_{n-1}. Recall that by Proposition 2.2.8, we may consider only left-most representatives. So let I be a left-most representative of some element in Q'_n. If I does not contain n, the element represented by I is one of the at most q'_{n-1} elements in the image of the map $Q'_{n-1} \to Q'_n$. Assume now I has an n, and consider the right-most appearance of n, so we can write

$$I = XnY$$

with $Y \in Q'_{n-1}$. If there are no 1's in X, then $X \in \mathrm{sh}_1(Q'_{n-1})$; this gives at most $q'_{n-1} \cdot q'_{n-1}$ possibilities for $[I]$. Otherwise, X contains 1, and we may now write $X = U1V$ with V not containing 1, i.e., we look for the right-most appearance of 1 in X. Now V may or may not contain n. In general, we can rewrite I as

$$I = U1WnZ$$

with W not containing 1 or n and Z either not containing n or of the form JnJ' with J not containing 1 and J' not containing n. Thus, varying Z will give only finitely many possibilities for $[I]$ by previous considerations. Since I is left-most and $1Wn$ is a connected subsequence of I, it is also left-most. Thus, $1Wn$ contains I_1^n as a possibly disconnected subsequence. Now by Proposition 7.3.2, Part A(n), we can find $J, K \in F_{n-2}$ with $1Wn \sim \mathrm{sh}_2(J)I_1^n \mathrm{sh}_1(K)$. By Proposition 7.3.2, Part C(n), we know that $U1Wn$ is equivalent to a word of the form

$$\mathrm{sh}_1(U')I_1^n \mathrm{sh}_1(K).$$

with $U' \in F_{n-1}$. Applying again the induction hypothesis, we see that there are also only finitely many possibilities for the word represented by $U1Wn$. This yields the claim. $\square$

7.4 Factorable Monoids with Complete Rewriting Systems

Recall that for the monoid Q_n, the Evaluation Lemma 2.2.2 of A. Heß holds, i.e., for any tuples $I, J \in F_n$ with $I \sim_Q J$, we know that we have the equality $f_I \equiv f_J$ in the graded sense. Unfortunately, only weak analogs are true for the monoid Q'_n. Yet, they are sufficient to provide some examples of complete rewriting systems. So we will show a suitable Evaluation Lemma for Q'_n. First, we will prove a variant of the Evaluation Lemma 2.2.2 for the original monoid Q_n.

Lemma 7.4.1. *Let $(M, \mathcal{E}, \eta)$ be a factorable monoid satisfying the stronger conditions $(xs)' = (x's)'$ and $\overline{xs} = \overline{x} \cdot \overline{x's}$ for any $x \in M$ and $s \in \mathcal{E}$. Then*

$$f_1 f_2 f_1 f_2 = f_2 f_1 f_2 = f_2 f_1 f_2 f_1$$

holds for all triples in this monoid. Recall that for a sequence $I = (i_s, \ldots, i_1) \in F_n$, we defined $f_I \colon M^{n+1} \to M^{n+1}$ to be the composition $f_{i_s} \circ f_{i_{s-1}} \circ \ldots \circ f_{i_1}$. If $I \sim_Q J$ and $(x_{n+1}, x_n, \ldots, x_1)$ is a tuple of elements $x_i \in \mathcal{E}$, then

$$f_I(x_{n+1}, \ldots, x_1) = f_J(x_{n+1}, \ldots, x_1).$$

(Note that this equality holds not only in the graded sense.)

Proof. Note we can rewrite the assumption $(xs)' = (x's)'$ and $\overline{xs} = \overline{x} \cdot \overline{x's}$ for any $x \in M$ and $s \in \mathcal{E}$ as $\eta_1 d_1 = d_2 \eta_1 d_1 \eta_2 \colon M \times \mathcal{E} \to M \times M$, where we use Notation 2.1.11. For the proof, we have first to show that $\eta_1 d_1 = d_2 \eta_1 d_1 \eta_2$ also holds for all pairs in $M \times M$. This works exactly as the proof of the graded equality by deleting the word "graded" (cf. Proposition 2.1.20 in [42] and its proof). The second part of the proof works by applying the same method to the proof of Lemma 2.2.5 in [42]. $\square$

Now we are ready to prove the Evaluation Lemma for Q_n'.

Lemma 7.4.2. *Let $(M, \mathcal{E}, \eta)$ be a factorable monoid satisfying the stronger conditions $(xs)' = (x's)'$ and $\overline{xs} = \overline{x} \cdot \overline{x's}$ for any $x \in M$ and $s \in \mathcal{E}$. If $I \sim J$ and $(x_{n+1}, x_n, \ldots, x_1)$ is a tuple of elements $x_i \in \mathcal{E}$, then*

$$f_I(x_{n+1}, \ldots, x_1) = f_J(x_{n+1}, \ldots, x_1).$$

Proof. It is enough to show this for the defining relation of $\sim$. So let $U, V \in F_n$ be two words in letters $1, \ldots, n$ and $1 \le k \le n$ such that $kU \sim_P Uk$, the letter k does not occur in U and such that $kUV \sim_Q UV$. We want to show that $f_k f_V$ and f_V are equal evaluated on each $(x_{n+1}, x_n, \ldots, x_1)$. We have only to show that under these conditions, $f_V(x_{n+1}, x_n, \ldots, x_1)$ is stable at the position k. Note that since k does not occur in U, the only possibility for $kU \sim_P Uk$ to be true is that neither $k+1$ nor $k-1$ occur in U. So applying f_U does not affect the letters in the k-th and the $k+1$-st place! Hence, in this places, $f_U f_V(x_{n+1}, x_n, \ldots, x_1)$ and $f_V(x_{n+1}, x_n, \ldots, x_1)$ have equal entries. Thus, the same holds for $f_k f_U f_V(x_{n+1}, x_n, \ldots, x_1)$ and $f_k f_V(x_{n+1}, x_n, \ldots, x_1)$. But $kUV \sim_Q UV$, so we can apply the Evaluation Lemma 7.4.1, which now holds in the proper and not only in the graded sense. $\square$

This gives us a (quite restrictive) sufficient condition for the completeness of the associated rewriting system.

Theorem 7.4.3. *Let $(M, \mathcal{E}, \eta)$ be a factorable monoid satisfying the stronger conditions $(xs)' = (x's)'$ and $\overline{xs} = \overline{x} \cdot \overline{x's}$ for any $x \in M$ and $s \in \mathcal{E}$. Then the associated string rewriting system is complete.*

Proof. This is basically proven in [42], Lemma 3.4.10. The only point to take care of is the proof of Lemma 3.4.9 in [42], which uses Proposition 3.4.4 of [42]. Although this proposition is not true in full generality, in our situation Corollary 7.3.3 and Lemma 7.4.2 guarantee that the statement of Proposition 3.4.4 of [42] holds under our stronger assumptions. This completes the proof. $\qquad\square$

Remark 7.4.4. *Proposition 3.4.4 of [42], stating a stronger version of our Lemma 7.4.2, is unfortunately wrong. It cannot be proved without further assumptions, as the counterexample in Section 7.1 shows. Thus, the proof of noetherianity for the rewriting system associated to a factorable monoid cannot be fixed in general. Yet, it can be fixed in special cases, e.g. as in Theorem 7.4.3 above. Note that the first half of Section 3.4 of [42] and parts of present section were developed jointly by A.Heß and the current author.*

Corollary 7.4.5. *The rewriting system on a left locally Gaussian monoid associated to a factorability structure from Theorem 4.2.5 is complete.*

Proof. Let M be a left locally Gaussian monoid and $\mathcal{E}$ a generating set closed under left least common multiples and left complements. We have to show that $(xs)' = (x's)'$ and $\overline{xs} = \overline{x} \cdot \overline{x's}$ for any $x \in M$ and $s \in \mathcal{E}$. Recall that the factorability structure was defined via $\eta(x) = (x_p \dots x_2, x_1)$ for $\mathrm{NF}(x) = x_p \dots x_2 x_1$ for $x \neq 1$ and $\eta(1) = (1, 1)$, where NF denotes the $\mathcal{E}$-normal form. Recall furthermore that by Corollary 4.2.1, x_1 can be characterized as the greatest right divisor of x lying in $\mathcal{E}$.

So we have to compare the greatest right divisors of xs and $x_1 s$ lying in $\mathcal{E}$. Call them a and b, respectively. Observe that since $xs \succeq x_1 s$, we can conclude that $a \succeq b$. Since $N_{\mathcal{E}}(x_1 s) \leq 2$, we know that the $\mathcal{E}$-normal form of $x_1 s$ has length at most 2. So write $x_1 s = tb$ with $t \in \mathcal{E} \cup \{1\}$. Furthermore, we can write $a = cb$ with $c \in \mathcal{E} \cup \{1\}$ since $\mathcal{E}$ is closed under left complements.

Now if $t = 1$, then $a = cx_1 s$ and there is a $d \in M$ such that

$$xs = x_p \dots x_1 s = da = dcx_1 s,$$

so that cx_1 right-divides x by right-cancellativity. Now observe that since $s, a \in \mathcal{E}$, also $cx_1 \in \mathcal{E}$ since $\mathcal{E}$ is closed under left complements. Thus, cx_1 is a right divisor of x lying in $\mathcal{E}$ and by definition of x_1, it has to right-divide x_1. So we conclude $c = 1$ and thus $a = b$ in the case $t = 1$.

Assume now $t \neq 1$. Since b is the greatest right-divisor of $x_1 s$ lying in $\mathcal{E}$, it has to be right-divisible by the right divisor $s \in \mathcal{E}$ of $x_1 s$; so there is a $u \in \mathcal{E}$ such that $b = us$. By the same argument, there is a $v \in \mathcal{E}$ so that $a = vs$ holds. We can again find a $d \in M$ such that $xs = da$. Inserting $a = vs$ and cancelling on the right, we obtain $x = dv$. Hence, there is a $w \in E$ so that $x_1 = wv$, by the definition of x_1. This implies that $x_1 s = wvs = wa$. By the definition of b, this shows that a is a right divisor of b. Since we already had $a \succeq b$, the statement $a = b$ follows. This proves $(xs)' = (x's)'$.

Since $xs = \overline{xs} \cdot (xs)'$ on the one hand, and, on the other hand,

$$xs = \overline{x}x's = \overline{x} \cdot \overline{x's} \cdot (x's)',$$

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and since we have already shown $(xs)' = (x's)'$, right-cancellativity implies $\overline{x} \cdot \overline{x's} = \overline{xs}$. This completes the proof since now we are in the situation to use Theorem 7.4.3.

$\square$

Remark 7.4.6. *Note that the corollary above in particular provides a complete rewriting system on each Artin monoid. Besides the finite-type Artin monoids ([41]), some other complete rewriting systems for subclasses of Artin monoids were known before (cf. [3]).*

In the case of Garside groups, there are already complete rewriting systems describing them due to S. Hermiller and J. Meier ([41]).

Remark 7.4.7. *Let M be a Garside monoid with a Garside element Δ. Let G be the group of fractions of M, and let $\mathcal{D}$ be the set of left-divisors of Δ except for 1. Then Lemma 7.1.1 allows us to associate a rewriting system to the factorability structure on $(G, \mathcal{D} \cup \mathcal{D}^{-1})$ described in Section 4.6. This rewriting system is exactly the second rewriting system R_2 for Garside groups in [41]. There, this rewriting system is shown to be complete. By this argument, we know that the rewriting systems associated to factorability structures in Garside groups are complete.*

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Zusammenfassung

Diese Dissertation beschäftigt sich mit kombinatorischen Eigenschaften von Gruppen und Monoiden im Hinblick auf ihre Homologie und ihre klassifizierende Räume.

C.-F. Bödigheimer und B. Visy haben bei den Berechnungen der Homologie von Modulräumen festgestellt, dass die symmetrischen Gruppen $\mathfrak{S}_n$ eine Zusatzstruktur tragen, die für jede Permutation eine bevorzugte Zerlegung in Transpositionen liefert. Sie haben die Eigenschaften dieser Zusatzstruktur zum Begriff der Faktorabilität axiomatisiert, der einen kleineren Komplex als den Bar-Komplex zur Berechnung der Homologie dieser Gruppen liefert.

Ist eine Gruppe mit einem festen Erzeugendensystem vorgegeben, so ist die Faktorabilitätsstruktur eine Abbildung, die von jedem Element der Gruppe einen Erzeuger abspaltet, sodass gewisse Verträglichkeitsbedingungen im Bezug auf die Wortlänge und auf die Multiplikation erfüllt sind. Die Existenz solcher Abbildungen hängt mitunter von dem gewählten Erzeugendensystem ab. So ist beispielsweise die symmetrische Gruppe mit dem Erzeugendensystem aller Transpositionen faktorabel, aber für das Erzeugendensystem der einfachen Transpositionen kann gezeigt werden, dass es keine Faktorabilitätsstruktur zulässt. Dies ist ein Spezialfall des Satzes von M. Rodenhausen, der besagt, dass es für jede faktorable Gruppe bezüglich des gewählten Erzeugendensystems eine Präsentation mit Relationen der Länge höchstens 4 geben muss. Da der Beweis dieses Theorems bisher unveröffentlicht ist, nehmen wir ihn im Abschnitt 2.3 auf.

R. Wang([61]) und A. Heß([42]) haben die Definition der Faktorabilität auf Monoide verallgemeinert. A. Heß hat mit Hilfe der diskreten Morsetheorie gezeigt, dass auch dieser Begriff einen kleinen Komplex zur Berechnung der Homologie faktorabler Monoide liefert. Das ist unter anderem deswegen von Interesse, da man manchmal auf die Homologie verwandter Gruppen schließen kann.

Ein wichtiges Ziel dieser Dissertation ist es, neue Familien von Beispielen für Faktorabilität vorzustellen. Eine wichtige Quelle für Beispiele stellt die Garside-Theorie dar. Eins der Hauptresultate dieser Arbeit ist das folgende Theorem:

Theorem. *(vgl. 4.2.5) Sei M ein linkes lokal-gaußsches Monoid, und $\mathcal{E}$ ein Erzeugendensystem, das unter linken kleinsten gemeinsamen Vielfachen und unter linken Komplementen abgeschlossen ist. Dann kann $(M, \mathcal{E})$ mit einer Faktorabilitätsstruktur versehen werden.*

Die Garside-Theorie wurde von P. Dehornoy begründet (z.B. [24]); zu weiteren wichtigen Entwicklungen zählen Artikel von P. Dehornoy und Y. Lafont ([28]) und von P. Dehornoy und L. Paris ([29]). Heutzutage gibt es viele Varianten der Garside-Theorie. Eine ausführliche Abhandlung zu diesem Thema ist in [27] zu finden. Zu den wichtigsten Beispielen für die Garside-Theorie zählen die Artin-Gruppen, eine Verallgemeinerung von den Zopfgruppen.

Mit Artin-Gruppen beschäftigt sich auch ein weiterer Teil dieser Arbeit. Artin-Gruppen sind in der gleichen Weise zu allgemeinen Coxeter-Gruppen assoziiert wie die Zopfgruppen zu den symmetrischen Gruppen. Die Artin-Gruppen, die zu den endlichen

Coxeter-Gruppen assoziiert sind, sind recht gut verstanden. Unter anderem weiß man, dass ein gewisses Hyperebenenkomplement in der Standarddarstellung der zugehörigen Coxeter-Gruppe ein Modell für den klassifizierenden Raum der Gruppe liefert. Man kann analoge Hyperflächenkomplemente auch im Falle der unendlichen Coxeter-Gruppen betrachten; es ist bekannt, dass deren Fundamentalgruppe genau die zugehörige Artin-Gruppe ist. Man kann sich also fragen, ob diese Räume ebenfalls Modelle für die klassifizierenden Räume liefern. Dieses Problem ist im Allgemeinen ungelöst und ist der Gegenstand der $K(\pi, 1)$-Vermutung:

Vermutung ($K(\pi, 1)$-Vermutung). *Sei G eine Artin-Gruppe und W die zugehörige Coxeter-Gruppe. Sei V die Standarddarstellung von W durch Isometrien, und $R \subset W$ bezeichne die Menge der Elemente r, die auf eine Reflektion an einer Hyperebene $H_r \subset V$ abgebildet werden. Dann ist der Quotient des komplexen Hyperebenenkomplements $V \otimes \mathbb{C} \setminus \bigcup_{r \in R} H_r \otimes \mathbb{C}$ nach der W-Wirkung ein $K(G, 1)$-Raum.*

Die Vermutung ist bereits in vielen Fällen bewiesen: Zum Beispiel von P. Deligne in [30] für den Fall der endlichen Coxeter-Gruppen, in [18] für die sogenannten Artin-Gruppen vom FC-Typ und in [40] für die Artin-Gruppen vom großen Typ. Weitere Informationen zu Artin-Gruppen und der $K(\pi, 1)$-Vermutung können beispielsweise in [16], [34], [17], [18], [39], [54] gefunden werden.

Wir werden eine alternative Formulierung der $K(\pi, 1)$-Vermutung aus [18] nutzen, um einen neuen Beweis des kürzlich erschienenen Theorems von N.Dobrinskaya ([33]) zu liefern, das die $K(\pi, 1)$-Vermutung mit dem klassifizierenden Raum des zugehörigen Monoids in Beziehung setzt (s. auch Abschnitt 5.3):

Theorem. *(vgl. [33]) Ist G eine Artin-Gruppe und M das zugehörige Artin-Monoid, so ist die natürliche Abbildung $BM \to BG$ genau dann eine Homotopieäquivalenz, wenn die $K(\pi, 1)$-Vermutung für die Gruppe G wahr ist.*

Der in dieser Arbeit präsentierte Beweis unterscheidet sich deutlich von dem ursprünglichen und basiert auf diskreter Morse-Theorie, einem Analogon der gewöhnlichen Morse-Theorie für CW-Komplexe. Es gibt viele verschiedene Varianten diskreter Morse-Theorie; für unsere Zwecke scheint die Version von E. Batzies ([4]) am besten geeignet.

In dieser Arbeit werden außerdem neue Familien von faktorablen Gruppen erarbeitet: Die orthogonalen Gruppen und die Coxeter-Gruppen der B-Reihe, jeweils mit dem Erzeugendensystem aller Reflektionen. Die Einschränkung der Faktorabilitätsstruktur auf der orthogonalen Gruppe $O(n)$ auf die Untergruppe $\mathfrak{S}_n$ stimmt mit der Faktorabilitätsstruktur von Visy und Bödigheimer überein.

Zuletzt beschäftigen wir uns mit der Frage nach der Verbindung zwischen faktorablen Monoiden und vollständigen Neuschreibsystemen, die zuerst von A. Heß gestellt wurde. Ein Neuschreibsystem ist einfach eine Monoidpräsentation, in der jede Relation mit einer Richtung versehen ist, in die sie angewendet werden kann. Ein vollständiges Neuschreibsystem erlaubt es, jedes Wort in endlich vielen Schritten durch die Anwendung der Neuschreibregeln in eine Normalform zu bringen. Ein vollständiges Neuschreibsystem

auf einem Monoid erlaubt auch gewisse Aussagen über die Homologie dieses Monoids. Jedem faktorablen Monoid ist ein Neuschreibsystem zugeordnet, das jedoch nicht immer vollständig ist, wie an einem Beispiel gezeigt wird. Jedoch kann man in einigen Fällen zeigen, dass das Neuschreibsystem vollständig ist. Wir beweisen das folgende Theorem im Abschnitt 7.4.

Theorem. *Das Neuschreibsystem, das der Faktorabilitätsstruktur auf einem linken lokalgaußschen Monoid zugeordnet ist, ist stets vollständig.*