Exercises in Geometry II

University of Bonn, Summer Semester 2018
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Sheet 9

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1. A Generalization of Bonnet-Myers [4 points]

Let (M, g) be a complete connected n-dimensional Riemannian manifold and suppose that there exist constants $a > 0$ and $c > 0$ such that, for all pairs of points $p, q \in M$ and for all minimizing geodesics $\gamma(s)$, parametrized by arclength, joining p and q, we have

$$\operatorname{Ric}(\gamma'(s)) \geq a + \frac{df}{ds}$$

along γ, for a function $f(s)$ with $|f(s)| \leq c$ along γ.
Show that M is compact.
Hint: Calculate an estimate for the diameter of M.

2. The second variation of the energy for a non-proper variation [4 points]

Let $\gamma : [0, a] \to M$ be a geodesic in a complete connected Riemannian manifold (M, g) and let $f : (-\varepsilon, \varepsilon) \times [0, a] \to M$ be a variation of γ that is not necessarily proper. Let V be the variation field and E be the energy function of the variation f.
Show that

$$\begin{aligned}\frac{1}{2}E''(0) = &-\int_0^a \Big\langle V(t), \frac{D^2V}{dt} + R(V, \frac{d\gamma}{dt})\frac{d\gamma}{dt}\Big\rangle dt \\ &- \sum_{i=1}^{k} \Big\langle V(t_i), \frac{DV}{dt}(t_i^+) - \frac{DV}{dt}(t_i^-)\Big\rangle \\ &- \Big\langle \frac{D}{ds}\frac{\partial f}{\partial s}, \frac{d\gamma}{dt}\Big\rangle(0,0) + \Big\langle \frac{D}{ds}\frac{\partial f}{\partial s}, \frac{d\gamma}{dt}\Big\rangle(0,a) \\ &- \Big\langle V(0), \frac{DV}{dt}(0)\Big\rangle + \Big\langle V(a), \frac{DV}{dt}(a)\Big\rangle,\end{aligned}$$

where $\{t_i\}_{1 \leq i \leq k}$ are the points where V is not smooth and

$$\begin{aligned}\frac{DV}{dt}(t_i^+) &= \lim_{t \searrow t_i} \frac{DV}{dt}, \\ \frac{DV}{dt}(t_i^-) &= \lim_{t \nearrow t_i} \frac{DV}{dt}.\end{aligned}$$

3. O'Neill's formula [4 points]

Let $f\colon (\tilde{M}, \tilde{g}) \to (M, g)$ be a Riemannian submersion. Let $\widetilde{\sec}$ and $\sec$ be the sectional curvatures of $\tilde{g}$ and g, respectively.
Show that for all horizontal vector fields $X, Y \in \Gamma(T\tilde{M})$, satisfying $|X| = |Y| = 1$ and $\tilde{g}(X, Y) = 0$, we have the identity

$$\widetilde{\sec}(X, Y) = \sec(f_*X, , f_*Y) - \frac{3}{4}|[X, Y]^V|^2.$$

Recall, that $[X, Y]^V$ denotes the vertical component of $[X, Y]$.

4. The sectional curvature on $\mathbb{C}\mathrm{P}$ [4 points]

For any $n \in \mathbb{N}$ we view the unit round sphere S^{2n+1} as a subset of $\mathbb{C}^{n+1}$. The circle S^1 acts on S^{2n-1} via componentwise multiplication, i.e. for $\theta \in S^1$ and $(z_0, \dots, z_n) \in S^{2n+1}$,

$$\theta \cdot (z_0, \dots, z_n) \mapsto (\theta \cdot z_0, \dots, \theta \cdot z_n).$$

It is well-known that the quotient S^{2n+1}/S^1 is diffeomorphic to $\mathbb{C}\mathrm{P}^n$.

a) Show that the standard round metric $\tilde{g}$ on S^{2n-1} descends to a well-defined metric g on $\mathbb{C}\mathrm{P}^n$ such that the quotient map $f : (S^{2n+1}, \tilde{g}) \to (\mathbb{C}\mathrm{P}^n, g)$ is a Riemannian submersion.

b) Show that the sectional curvature of $\mathbb{C}\mathrm{P}^n$ satisfies

$$1 \leq \sec(X, Y) \leq 4,$$

for all $X, Y \in \Gamma(T\mathbb{C}\mathrm{P}^n)$ with $|X| = |Y| = 1$ and $g(X, Y) = 0$. Are these bounds sharp?

Due on Monday, July 9.

Homepage of the lecture: `https://www.math.uni-bonn.de/people/galazg/`