Exercises in Geometry II

University of Bonn, Summer Semester 2018
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Sheet 8

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1. Vector field relations in a Riemannian submersion [4 points]

Let $f : (\tilde{M}, \tilde{g}) \to (M, g)$ be a Riemannian submersion. Recall, from the last exercise sheet, the decomposition of the tangent space

$$T\tilde{M}_x = H_x + V_x$$

into the horizontal and the vertical space, for each $x \in \tilde{M}$. A smooth vector field on $\tilde{M}$ is called *horizontal* (resp. *vertical*) if it takes values in the horizontal (resp. vertical) space at each point. In the following, $\tilde{X}$ always denotes the horizontal lift of a vector field $X \in \Gamma(TM)$.

a) For any smooth vector fields $X, Y \in \Gamma(TM)$ show that

$$\tilde{g}(\tilde{X}, \tilde{Y}) = f^*g(X, Y),$$
$$[\tilde{X}, \tilde{Y}]^H = \widetilde{[X, Y]},$$
$$[\tilde{X}, W] \text{ is vertical if } W \text{ is vertical.}$$

b) Let $\tilde{\nabla}$ and ∇ denote the Levi-Civita connections of $\tilde{g}$ and g respectively. Show that for any smooth vector fields $X, Y \in \Gamma(TM)$,

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y} = \widetilde{\nabla_X Y} + \frac{1}{2}[\tilde{X}, \tilde{Y}]^V.$$

Hint: Use the Koszul formula.

2. Horizontal geodesics [4 points]

Let $f : (\tilde{M}, \tilde{g}) \to (M, g)$ be a Riemannian submersion. Show that, if a geodesic $\tilde{\gamma}$ of $\tilde{M}$ is horizontal at some point, then it is horizontal everywhere and $\gamma = f \circ \tilde{\gamma}$ is a geodesic in M.

3. Group actions and Riemannian submersions [4 points]

Let $(\tilde{M}, \tilde{g})$ be a Riemannian manifold and $f : \tilde{M} \to M$ be a submersion. Assume that a compact Lie group G acts smoothly on $\tilde{M}$ by isometries so that $f \circ \varphi = f$ for all $\varphi \in G$. Suppose further that G acts transitively on each fiber $\tilde{M}_p := f^{-1}(p)$, $p \in M$.
Show that there is a unique Riemannian metric g on M such that f is a Riemannian submersion.
Hint: Show that $\varphi_* V_x = V_{\varphi(x)}$ for all $\varphi \in G$.

4. A metric on the tangent bundle [4 points]

Construct a Riemannian metric $\tilde{g}$ on the tangent bundle M of a Riemannian manifold (M, g) such that the projection $\pi : (TM, \tilde{g}) \to (M, g)$ is a Riemannian submersion and the metric $\tilde{g}$ restricted to the tangent spaces is the Euclidean metric.

Due on Monday, July 2.

Homepage of the lecture: `https://www.math.uni-bonn.de/people/galazg/`