Exercises in Geometry II

University of Bonn, Summer Semester 2018
Dozent: PD Dr. Fernando Galaz-Garcia
Assistant: Saskia Roos
Sheet 6

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1. The index form [4 points]

Let $\gamma : [0, l] \to (M, g)$ be a unit speed geodesic. Recall that for any pair of proper normal vector fields V, W along γ the index form is given by

$$I(V, W) = \int_a^b (\langle D_t V, D_t W\rangle - \langle R(V, \dot\gamma)\dot\gamma, W\rangle) dt.$$

Show that the index form can also be written as

$$I(V, W) = -\int_a^b \langle D_t^2 V + R(V, \dot\gamma)\dot\gamma, W\rangle dt - \sum_{i=1}^{k} \langle \Delta_i D_t V, W(a_i)\rangle,$$

where $\{a_i\}$ are the points where V is not smooth and $\Delta_i D_t V$ is the jump in $D_t V$ at $t = a_i$.

2. The null space of the index form [4 points]

Recall the index form from the previous exercise.

a) Show that the null space of I is exactly the set of Jacobi fields along γ vanishing at $\gamma(a)$ and $\gamma(b)$. Specifically, V is a Jacobi field if and only if $I(V, W) = 0$ for all W.

b) Show that I has a nontrivial null space if and only if $\gamma(a)$ is conjugate to $\gamma(b)$. The dimension of the null space is the order of the conjugate point $\gamma(b)$.

3. The injectivity radius

a) Is the injectivity radius of a complete, compact Riemannian manifold (M, g) always positive? Prove this or provide a counter example.

b) Give an example of a complete Riemannian manifold (M, g) whose injectivity radius is zero.

4. Orientable surfaces of constant negative curvature [4 points]

Show that every compact orientable surface of genus $g \geq 2$ without boundary admits a metric of constant negative curvature.
Hint: Use the Gauß–Bonnet Theorem to show that each such surface can be realized as the identification space obtained from a regular polygon P with $4g$ geodesic sides making an angle of $\pi/2g$ in the hyperbolic plane H^2, whose sides are glued together via isometries, i.e. the universal cover is H^2.

Due on Tuesday, June 19.

Homepage of the lecture: `https://www.math.uni-bonn.de/people/galazg/`