Journal of Knot Theory and Its Ramifications, Vol. 9, No. 5 (2000) 611–615

ON WHITE'S FORMULA

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Received 1 September 1999

Abstract

A smooth knot is non-trivial precisely if it cannot be unlinked from every small deformation of itself. White's formula expresses the linking number of a representative of a knot and a deformation of it as a sum of two, not necessarily integral, terms – a writhe and a twist. An elementary proof is given of this formula. A byproduct of the proof is an easy derivation of the expression for the writhe of the knot in terms of standard invariants of the curve and the writhe of the knot diagram obtained by projection onto a plane.

1 Statement of White's Formula

Let K be a smooth knot in $\mathbb{R}^3$ and $k : S \to \mathbb{R}^3$, where S is a circle, be the smooth, injective, arc length preserving map that parametrises K. Let v be a smooth non-zero vector field defined on a neighbourhood of K. In White's formula one considers the case where at each point $k(s)$ of K the vector $v(s)$ has unit length and is perpendicular to the unit tangent $k'(s)$ of K. One further supposes that the scale has been chosen sufficiently small so that the intervals in $\mathbb{R}^3$ that join $k(s)$ to $k(s)+v(s)$ are disjoint for distinct $s \in S$. These intervals form a ribbon, as s varies, with edges K and K_v (say).

The **linking number** of K and K_v is given by the well-known formula of Gauss

$$Lk(K, K_v) = -\frac{1}{4\pi} \iint_{S\times S} g_1^*(w) \ ,$$

where $g_1 : S \times S \to \mathbb{R}^3 \setminus \{0\}$ is given by $g_1(s,t) = k(s) - k(t) - v(t)$ and where w is the 2-form $(x_1^2+x_2^2+x_3^2)^{-\frac{3}{2}}(x_1 dx_2{\wedge} dx_3 - x_2 dx_1{\wedge} dx_3 + x_3 dx_1{\wedge} dx_2)$ on $\mathbb{R}^3 \setminus \{0\}$. The **twisting number** $Tw(K,v)$ is defined to be

$$Tw(K,v) = \frac{1}{2\pi} \int_S (k'(s) \times v(s)).v'(s) ds \ .$$

Using the Serret-Frenet equations one can rewrite this definition as the formula, which first appeared in [3],

$$Tw(K,v) = \frac{1}{2\pi}\int \tau ds + d(K,v)$$

where τ is the torsion of the curve K and the integer $d(K,v)$ is the number of revolutions about the origin that v makes in the Frenet frame as $k(s)$ moves once around K. The **writhing number** $Wr(K)$ is defined to be

$$Wr(K) = -\frac{1}{4\pi}\iint_{S\times S\setminus\Delta} g_o^*(w)\ ,$$

where $g_0(s,t) = k(s) - k(t)$ and Δ is the diagonal subset $\{(s,s) : s \in S\}$ of $S \times S$. A more elementary formula for $Wr(K)$ is mentioned in the final remark of §2.

White's formula is the theorem that

$$Lk(K, K_v) = Wr(K) + Tw(K,v).$$

Here, although $Lk(K, K_v)$ is always an integer, in general $Wr(K)$ and $Tw(K,v)$ are not integers.

A very elegant proof by White [11] of the formula exploits the algebraic geometric concept of a blow-up. A fluid dynamics proof based on the invariance of helicity may be found in [7]. This paper also contains a historical survey of the theorem and mention of its relevance in applied mathematics. White's formula is of interest to biologists ([1,6,11]) and it serves a purpose to record a proof, that is much less sophisticated than [7,11] and much shorter than previous elementary proofs by Călugăreanu, Pohl and others ([3,4,5,9,10,12]). For recent generalisations of White's formula the reader is referrred to [2].

2 A Proof of White's Theorem

Consider the region $Q_\varepsilon = \{(s,t,u) \in S \times S \times I : |s-t|^2 + u^2 \geq \varepsilon^2\}$ in $S \times S \times I$. The boundary of Q_ε, for ε sufficiently small, is $\partial Q_\varepsilon = B_\cup F_\cup N$, where

$$\begin{aligned} B &= \{(s,t,1) : s \in S, t \in S\} \\ F &= \{(s,t,0) : s \in S, t \in S, |s-t| \geq \varepsilon\} \\ N &= \{(s, s-\varepsilon\cos\theta, \varepsilon\sin\theta) : s \in S, 0 \leq \theta \leq \pi\}. \end{aligned}$$

Define $g : Q_\varepsilon \to \mathbb{R}^3\setminus\{0\}$ by $g(s,t,u) = k(s) - k(t) - uv(t)$.

The 2-form w on $\mathbb{R}^3 \setminus \{0\}$ is closed, since $dw = -3r^{-4}dr_\wedge(x_1dx_{2\wedge}dx_3 - x_2dx_{1\wedge}dx_3 + x_3dx_{1\wedge}dx_2) + 3r^{-3}dx_{1\wedge}dx_{2\wedge}dx_3 = 0$, where $r^2 = x_1^2 + x_2^2 + x_3^2$ (and so $rdr = x_1dx_1 + x_2dx_2 + x_3dx_3$). If we orient N by $ds_\wedge d\theta$, then

$$\begin{aligned}\int_B g^*w - \int_F g^*w - \int_N g^*w &= \int_{\partial Q_\varepsilon} g^*w \\ &= \int_{Q_\varepsilon} d(g^*w) \\ &= \int_{Q_\varepsilon} g^*(dw) \\ &= 0\,,\end{aligned}$$

where the second equality is obtained from Stokes' theorem. Since $\int_B g^*w = -4\pi Lk(K, K_v)$ and $\lim_{\varepsilon\to 0}\int_F g^*w = -4\pi Wr(K)$, to establish White's formula it suffices to show that $\lim_{\varepsilon\to 0}\int_N g^*w = -4\pi Tw(K,v)$.

To show this we first evaluate g^*w. By definition g^*w is found by substituting $x_i = k_i(s) - k_i(t) - uv_i(t)$ and $dx_i = k_i'(s)ds - k_i'(t)dt - uv_i'(t)dt - v_i(t)du$, for $i = 1, 2, 3$, into the definition of w. Rearranging the terms we find that

$$\begin{aligned}-g^*w &= \frac{1}{|g|^3}[g, k'(s), k'(t) + uv'(t)](ds_\wedge dt + \frac{1}{|g|^3}[g, k'(s), v(t)]ds_\wedge du \\ &\quad + \frac{1}{|g|^3}[g, v(t), k'(t) + uv'(t)]dt_\wedge du \qquad (2.1)\end{aligned}$$

where $g = g(s,t,u) = k(s) - k(t) - uv(t)$ and the notation $[a,b,c]$ is used for the scalar triple product $a.(b\times c)$.

For $(s,t,u) \in N$ we have $t = s - \varepsilon\cos\theta$, where ε is small. By Taylor's theorem $k(t) = \sum_{i=0}^{\infty}(-\varepsilon\cos\theta)^i k^{(i)}(s)$. Similarly expand $v(t)$, and hence $g(t), k'(t), v'(t)$, about s and substitute these expressions into (2.1). On N, $u = \varepsilon\sin\theta$ and writing $du = \varepsilon\cos\theta d\theta$ and $dt = ds + \varepsilon\sin\theta d\theta$ we now express g^*w in terms of s, θ (and ε) only. The reader can check that apart from $0(\varepsilon)$ terms the three summands on the righthand side of (2.1) contribute

$$(-[v,k',k'']\sin^2\theta\cos\theta + [v,k',v']\sin^3\theta)ds_\wedge d\theta,$$

$$(-[v,k',v']\sin\theta\cos^2\theta + \frac{1}{2}[k'',k',v]\cos^3\theta - [v',k',v]\sin\theta\cos^2\theta)ds_\wedge d\theta$$

$$\text{and } ([k',v,k'']\cos^3\theta - [k',v,v']\cos^2\theta\sin\theta - [v,v',k']\sin\theta\cos^2\theta$$
$$+\frac{1}{2}[k'',v,k']\cos^3\theta - [v',v,k']\sin\theta\cos^2\theta)ds_\wedge d\theta$$

respectively, where v, v', k, k', k'' are all to be evaluated at s. To evaluate $\int_N g^*(w)$ we integrate over θ first. This gives

$$\int_N g^*(w) = -\int_S 2[v'(s), k'(s), v(s)]ds + 0(\varepsilon),$$

and hence

$$\lim_{\varepsilon \to 0} \int_N g^*(w) = -4\pi Tw(K, v).$$

We conclude with two remarks. The first disposes of the question inherently raised by the last sentence of §3 of [7].

Remark 1 Even when Lk(K,K_v) = 0, the knots K and K_v are necessarily linked if K represents a non-trivial knot. To see this suppose the contrary, so one can find a 3-ball B in $\mathbb{R}^3$ such that $K_v \subset B$ and $K \subset \mathbb{R}^3 \backslash B$. By the compactness of B and K one can thicken K to a tubular neighbourhood TK of K such that $TK \subset \mathbb{R}^3 \backslash B$. Since B is contractible K_v bounds a (possibly singular) 2-disc D in B. Let K_v^1 be the curve on TK in which the boundary of TK intersects the ribbon in §1 with edges K and K_v. Then K_v^1 is the boundary of the disc consisting of the union of D and the annulus of the ribbon between K_v^1 and K_v. No point of K_v^1 is a singular point of this disc. By Dehn's Lemma [8] K_v^1 bounds a non-singular disc in the 3-manifold $\overline{\mathbb{R}^3 \backslash TK}$, hence K_v^1 (and hence K) represents a trivial knot, and so we have a contradiction. Indeed the linkedness of K and K_v for all permissible v is equivalent to the non-triviality of the knot K. One can, however, of course easily construct representatives K of the trivial knot and permissible vector fields v such that K and K_v are linked.

Remark 2 Let c be a fixed unit vector in $\mathbb{R}^3$. Suppose that the projection of K in the direction c onto a plane has no singularities other than at most finitely many double points and let D denote the knot diagram obtained. Suppose the scale has been chosen so that the intervals in $\mathbb{R}^3$ that join $k(s)$ to $k(s) + c$ are disjoint for distinct $s \in S$. Let $\gamma(s)$ and $\beta(s)$ denote respectively the components in the direction c of the unit tangent and binormal to the curve K at $k(s)$ and let $\kappa(s)$ denote the curvature of K. Following the same method as the proof of White's theorem, except with the constant vector field c in the place of v, one obtains

$$Wr(K) = Lk(K, K_c) + \frac{1}{4\pi} \int_S \kappa(s)\beta(s)f(s)ds,$$

where $f(s) = \int_0^\pi (1 - 2\gamma(s)\cos\theta\sin\theta)^{-\frac{3}{2}} \cos\theta \sin^2\theta d\theta$. This gives an elementary formula for $Wr(K)$, since $Lk(K, K_c)$ is the writhe of the knot diagram D, i.e. the signed sum of the crossings of D.

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