Problem sheet 6 Rigid analytic geometry Winter term 2024/25

Let X be a spectral space. From Problem 3 on sheet 4 and Problems 3 and 4 on the previous sheet we know that X_{con} is a spectral space, and by Problem 4 on sheet 4 we know that $X \to X_{\text{con}}$ is a spectral map. Moreover:

Problem 1 (3 points). *Show that a subset of X is constructible if and only if it is clopen in X_{con}.*

Problem 2 (3 points). *Show that an open subset of X is quasicompact if and only if it is constructible.*

Problem 3 (2 points). *If $X \xrightarrow{f} Y$ is a spectral map, show that $X_{\text{con}} \xrightarrow{f} Y_{\text{con}}$ is spectral.*

Problem 4 (2 points). *If $X \xrightarrow{f} Y$ is a continuous map where Y is a spectral space, then $X_{\text{con}} \xrightarrow{f} Y_{\text{con}}$ is continuous if and only if f is spectral map.*

A point $x \in X$ is called a *specialization* of $\xi \in X$ and ξ a *generification* of x iff $x \in \overline{\{\xi\}}$.

Problem 5 (6 points). *Let Z be a subset of X which is closed in X_{con}. Show that its closure $\overline{Z}$ in X is the set of specializations of the elements of x.*

It follows that the closed subsets of X are the closed subsets of X_{con} which are closed under specialization while the open subsets of X are the open subsets of X_{con} which are closed under generification.

Definition 1. *A* Priestley space *is a pair $(Y, \preceq)$ where Y is a compact topological space and $\preceq$ a partial order on Y with excluded non-trivial equivalence (i. e., $a \preceq b$ and $b \preceq a$ implies $a = b$) and such that when $y \not\preceq \upsilon$, the following two conditions hold:*

- *There is a clopen $U \subseteq Y$ such that $y \in U$, $\upsilon \notin U$ and such that $u \in U$ and $u \preceq v$ implies $v \in U$.*
- *There is a clopen $V \subseteq Y$ such that $y \notin V$, $\upsilon \in V$ and such that $v \in V$ and $u \preceq v$ implies $u \in V$.*

A morphism $(Y, \preceq_Y) \to (Z, \preceq_Z)$ of Priestley spaces is a continuous map $Y \xrightarrow{f} Z$ such that $y \preceq_Y \upsilon$ implies $f(y) \preceq_Z f(\upsilon)$.

It is easy to see that the two conditions are indeed equivalent. The brief exposition of the theory of specral spaces given in these exercise sheets will culiminate in the result that for an arbitrary set Y, there is a bijection beween the sets of structures of a spectral space and of a Priestley space on Y. This bijection will be made in such a way that when it is applied on both Y and Z, a map $Y \xrightarrow{f} Z$ is spectral if and only if it is a morphism of Priestley speces.

As before let X be spectral space.

Problem 6 (2 points)**.** *Let $x \preceq \xi$ if and only if x is a specialization of ξ. Show that $(X, \preceq)$ is a Priestley space!*

It is an easy consequence of the definitions and the previous results that if Y is a spectral space and the map $X \xrightarrow{f} Y$ spectral, then $(X, \preceq_X) \xrightarrow{f} (Y, \preceq_Y)$ is a morphism of Priestley spaces.

Conversely, let $(Y, \preceq)$ be a Priestley space and $\mathfrak{B}$ the set of clopen subsets $\Omega \subseteq Y$ such that $y \in \Omega$ and $y \preceq v$ implies $v \in \Omega$. It is clear that $\mathfrak{B}$ is closed under finite intersections within Y, including the empty intersection Y. Thus it is a topology base for a topology on Y. Let Y^s be Y equipped with this topology. As the original topology is finer, for $\Omega \in \mathfrak{B}$ the quasi-compactness of Ω in Y implies its quasi-compactness in Y^s. This verifies one part of the definition of "spectral space" for Y^s.

Problem 7 (2 points)**.** *Show that Y^s is T_0.*

It can be shown that Y^s is spectral. By Problem 5, if one starts with a spectral space and equips it with the structure of a Priestley space by 6, then Y^s is the original "spectral" topology on Y. The opposite can also be shown.

Solutions should be e-mailed to my institute e-mail address (my second name (franke) at math dot uni hyphen bonn dot de) before Monday December 2.