Shape Optimization Under Uncertainty - A Stochastic Programming Perspective

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Outline

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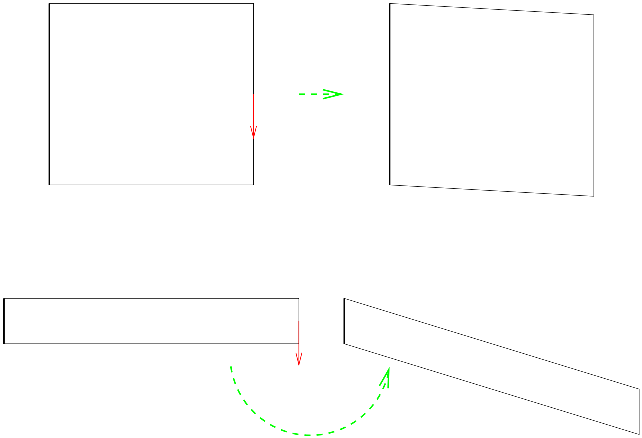
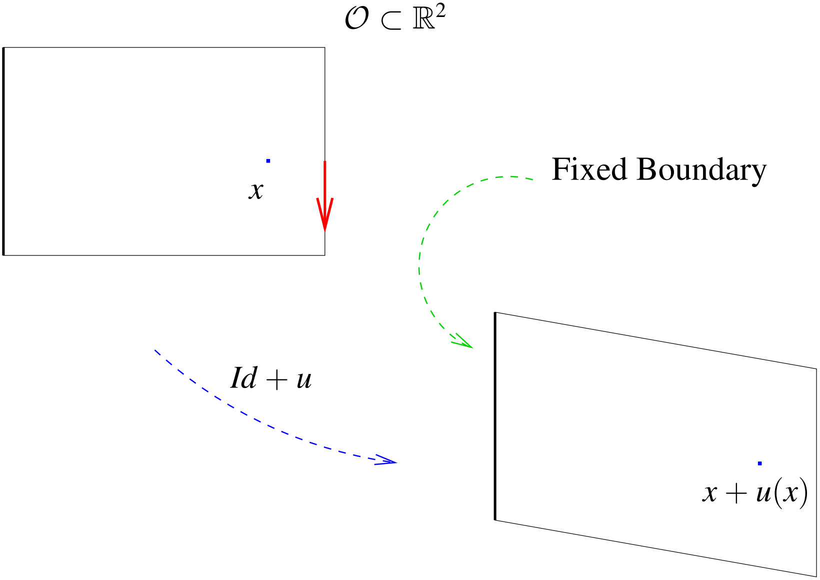
Deformations Depend on the Shape



Problem Setting



Linear Elasticity

- Elastic body $\mathcal{O} \subset \mathbb{R}^d$
- The boundary $\partial\mathcal{O}$ consists of two disjoint parts:

$$\partial\mathcal{O} = \Gamma_N \cup \Gamma_D, \ \Gamma_D \neq \emptyset$$

- Internal forces f
- External forces g

$\rightsquigarrow$ displacements u $\rightsquigarrow$ strain characterized by linearized strain tensor

$$e(u) = \frac{1}{2}(\nabla u + \nabla u^T), \ e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

Linear Elasticity

Elastic material behaves according to Hooke's law

$$A\xi = 2\mu\xi + \lambda(\mathrm{tr}\xi)\mathrm{Id}, \text{ for any symmetric matrix } \xi$$

$\mathcal{O}$ varying $\rightsquigarrow$ working domain D, contains all admissible shapes, $f \in L^2(D)^d$, $g \in H^1(D)^d$

PDE

$$\begin{cases} -\mathrm{div}(Ae(u)) = f & \text{in } \mathcal{O}, \\ u = 0 & \text{on } \Gamma_D, \\ (Ae(u))n = g & \text{on } \Gamma_N \end{cases}$$

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Composite Finite Elements

Standard FE:

- mesh has to resolve the structure of the domain
- therefore, min. dim. of FE space is directly linked to number and size of geometric details of the domain

More efficient: Composite FE (Developed by S. Sauter)

- allow coarse-level discretizations of PDEs on complicated domains
- principle idea: the shape of FE functions is hierarchically adapted to behavior of the solution $\rightsquigarrow$ discretization of problems with complicated structures with very few unknowns

Examples for Objective Functions

- Compliance

$$J(\mathcal{O}) = \int_{\mathcal{O}} f \cdot u \, \mathrm{d}x + \int_{\Gamma_N} g \cdot u \, \mathrm{d}s$$

- Least square error compared to target displacement

$$J(\mathcal{O}) = \left(\int_{\mathcal{O}} |u - u_0|^2 \, \mathrm{d}x \right)^{\frac{1}{2}}$$

Optimization Problem

$$\inf_{\mathcal{O} \in \mathcal{U}_{\mathrm{ad}}} J(\mathcal{O}) + \ell P(\mathcal{O})$$

existence of optimal shapes requires smoothness, geometrical or topological constraints

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Two-Stage Stochastic Linear Program

Information Constraint

decide $x \mapsto$ observe $z(\omega) \mapsto$ decide $y = y(x, z(\omega))$

$$\min_x \{c^T x + \min_y \{q^T y : \; Wy = z(\omega) - Tx, \; y \in Y\} \qquad : \; x \in X\}$$

$$\min_x \{c^T x + G(x, \omega) \qquad : \; x \in X\}$$

$\rightarrow$ looking for a minimal member in family of random variables $\{c^T x + G(x, \omega) : \; x \in X\}$

Risk-Neutral Setting

In this case, the random variables are ranked by their expectations.

$$\rightsquigarrow \min\{\mathbb{E}_\omega[c^T x + G(x,\omega)] : \; x \in X\}$$

Outline

General Objective Function

$$J(\mathcal{O}, u(\mathcal{O}, \omega)) = \int_{\mathcal{O}} j(u)\,\mathrm{d}x + \ell \int_{\partial\mathcal{O}} \mathrm{d}s,\ \ \mathcal{O} \in \mathcal{U}_{\mathrm{ad}}, \ell > 0$$

- $u = u(\mathcal{O}, \omega)$ is the solution of the PDE
- assume $j(.)$ is linear or quadratic and independent of ω

The Two Stages

- **First stage** Non-anticipative decision on $\mathcal{O}$ has to be taken
- The random forces $f(\omega), g(\omega)$ are observed
- **Second Stage** The variational formulation of elasticity, given $\mathcal{O}$ and ω, takes the role of the second-stage problem

Information constraint here

decide $\mathcal{O} \mapsto$ observe $f(\omega), g(\omega) \mapsto$ decide $u = u(\mathcal{O}, \omega)$

Variational Formulation of Elasticity

u also coincides with the minimizing element of

$$\inf\{E(\mathcal{O}, \varphi; \omega) : \varphi \in H^1(\mathcal{O})^d, \; \varphi = 0 \text{ on } \Gamma_D\},$$

$$E(\mathcal{O}, \varphi; \omega) = \int_{\mathcal{O}} \frac{1}{2} Ae(\varphi) \cdot\cdot\, e(\varphi) - f(\omega) \cdot \varphi \,\mathrm{d}x - \int_{\Gamma_N} g(\omega) \cdot \varphi \,\mathrm{d}s$$

Notation

$$A \cdot\cdot\, B = \mathrm{tr}(A^T B) = \sum_{i,j=1}^{d} A_{ij} B_{ij}$$

Two-Stage Shape Optimization Problem

$$\min\left\{\ell\int_{\partial\mathcal{O}}\mathrm{d}s+\int_{\mathcal{O}}j(u(\mathcal{O},\omega))\,\mathrm{d}x:\right.$$
$$\left.u(\mathcal{O},\omega)=\operatorname{argmin}\{E(\mathcal{O},\varphi;\omega):\ \varphi\in H^1(\Gamma_D)^d\},\mathcal{O}\in\mathcal{U}_{\text{ad}}\right\}$$

Direct Comparison with Linear Case

Linear 2-Stage Problem

$$\left.\begin{array}{l}\min\{\, F(x) + \mathbb{E}[G(\bar{y}(x,\omega)] \,:\; x \in X, \\ \bar{y}(x,\omega) \in \operatorname{argmin}\{G(y) \,:\; y \in Y(x,\omega)\}\end{array}\right\}$$

2-Stage Shape Optimization Program

$$\left.\begin{array}{l}\min\{\, \tilde{f}(\mathcal{O}) + \mathbb{E}[\tilde{g}(\mathcal{O}, \bar{u}(\mathcal{O},\omega))] \,:\, \mathcal{O} \in \mathcal{U}_{\mathrm{ad}}, \\ \bar{u}(\mathcal{O},\omega) \in \operatorname{argmin}\{\tilde{e}(\mathcal{O},u,\omega) \,:\; u \in H^1\}\end{array}\right\}$$

Structure of Random Forces

Now, the volume forces f and surface loads g are random with special structure:

$$\rightsquigarrow f = f(\omega),\ g = g(\omega),$$

- finitely many forces $f_1, \ldots, f_{K_1} \in L^2(\mathcal{O})^d$ and $g_1, \ldots, g_{K_2} \in H^1(\mathcal{O})^d$
- random coefficients $h_i^f(\omega), i = 1, \ldots, K_1$ and $h_i^g(\omega), i = 1, \ldots, K_2$ such that

$$f(\omega) = \sum_{i=1}^{K_1} h_i^f(\omega) f_i,\ g(\omega) = \sum_{i=1}^{K_2} h_i^g(\omega) g_i$$

Structure of Random Forces

- Additional requirement:

$$\sum_{i=1}^{K_1} h_i^f(\omega) = 1,\ \sum_{i=1}^{K_2} h_i^g(\omega) = 1,\ \forall\omega$$

- finitely many scenarios $\omega_i, i = 1, \ldots, S$ which occur with probabilities $\pi_i, i = 1, \ldots, S$

Lagrangian Functional

Consider Euler's equation as a constraint in the minimization problem and introduce the adjoint state ψ to construct a Lagrangian functional:

$$L(\mathcal{O}, \varphi, \psi; \omega) = J(\mathcal{O}, \varphi) + \mathrm{d}E(\mathcal{O}, \varphi, \omega; \psi)$$

First Variation

$$\mathrm{d}E(\mathcal{O}, \varphi, \omega; \psi) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} E(\mathcal{O}, \varphi + \varepsilon\psi; \omega)\Big|_{\varepsilon=0}$$

Optimality Conditions

The stationarity of the Lagrangian gives the optimality conditions:

$$\left\langle \partial_\varphi L(\mathcal{O}, \varphi^0, \psi^0; \omega), \phi \right\rangle = 0, \forall \phi \in H^1(\mathcal{O})^d,$$
$$\left\langle \partial_\psi L(\mathcal{O}, \varphi^0, \psi^0; \omega), \phi \right\rangle = 0, \forall \phi \in H^1(\mathcal{O})^d$$

- first condition $\rightsquigarrow$ adjoint problem
- second condition $\rightsquigarrow$ elasticity PDE

Adjoint Problem

The adjoint state p is the solution of the following problem:

$$\begin{cases} -\operatorname{div}(Ae(p)) = -j'(u) & \text{in } \mathcal{O}, \\ p = 0 & \text{on } \Gamma_D, \\ (Ae(p))n = 0 & \text{on } \Gamma_N \end{cases}$$

$\rightarrow$ will be needed for the shape derivative

Observations

- optimality conditions allow exactly one feasible solution $(\varphi_{\text{opt}}, \psi_{\text{opt}})$
- therefore, it's the optimal solution
- can be obtained by solving elasticity PDEs
- j was assumed to be at most quadratic $\rightsquigarrow j'$ is linear
- consequently, optimality conditions are linear in u, p, f, g and ϕ

Re-written Problem Formulation

Notation

$u^{(\mu,\nu)}, p^{(\mu,\nu)}$ denote the solutions of the elasticity problem $(P_{\mu,\nu})$ and the adjoint problem $(\hat{P}_{\mu,\nu})$, resp., with forces f_μ and g_ν, $\mu \in \{1, \dots, K_1\}, \nu \in \{1, \dots, K_2\}$

$$
\min\left\{ \begin{array}{l}
\ell \int_{\partial\mathcal{O}} \mathrm{d}s + \sum_{k=1}^{S} \pi_k \int_{\mathcal{O}} j(\bar{u}(\mathcal{O}, \omega_k))\, \mathrm{d}x : \\
\mathcal{O} \in \mathcal{U}_{\mathrm{ad}}, \\
\bar{u}(\mathcal{O}, \omega_k) := \sum_{\mu=1}^{K_1} h^f_\mu(\omega_k) \sum_{\nu=1}^{K_2} h^g_\nu(\omega_k) u^{(\mu,\nu)}, \\
\qquad k = 1, \dots, S, \\
u^{(\mu,\nu)} \text{ solves } (P_{\mu,\nu}), \\
\qquad \forall (\mu, \nu) \in \{1, \dots, K_1\} \times \{1, \dots, K_2\}
\end{array} \right\}
$$

Solution

Linearity and minimizing the expected value $\Rightarrow$ suffices to solve $K_1 + K_2$ PDEs, which is independent of the number of scenarios S.

Outline

Shape Derivative

Definition (Allaire et al)

The shape derivative of $J(\mathcal{O})$ at $\mathcal{O}$ is defined as the Fréchet derivative in $W^{1,\infty}(\mathbb{R}^d)^d$ at 0 of the mapping $\Theta \to J((\mathrm{Id} + \Theta)(\mathcal{O}))$, i.e.

$$J((\mathrm{Id} + \Theta)(\mathcal{O})) = J(\mathcal{O}) + \langle J'(\mathcal{O}), \Theta \rangle + o(\Theta)$$

with $\lim_{\Theta \to 0} \frac{|o(\Theta)|}{\|\theta\|} = 0$, where $J'(\mathcal{O})$ is a continuous linear form on $W^{1,\infty}(\mathbb{R}^d)^d$.

Form of Shape Derivative

The shape derivative is of the form

$$\left\langle \tilde{J}'(\mathcal{O}), \Theta \right\rangle = \int_{\partial\mathcal{O}} v\Theta \cdot n \, \mathrm{d}s,$$

with a function $v = v(\bar{u}_k, \bar{p}_k, n, H)$.

Domain Represented by Level-Set Function

$\mathcal{O}$ is described by means of a level-set function Φ in D:

$$\begin{cases} \Phi(x) = 0 & \Leftrightarrow x \in \partial\mathcal{O} \cap D, \\ \Phi(x) < 0 & \Leftrightarrow x \in \mathcal{O}, \\ \Phi(x) > 0 & \Leftrightarrow x \in (D \setminus \bar{\mathcal{O}}) \end{cases}$$

- normal n to $\mathcal{O}$ is $\frac{\nabla\Phi}{|\nabla\Phi|}$
- mean curvature H is given by div$n$

Shape Derivative in Level-Set Notation

- only variations in normal direction are interesting
- domain $\mathcal{O}$ is identified with level-set function Φ

$\rightsquigarrow$

$$\langle \tilde{J}'(\Phi), \vartheta \rangle = -\int_{[\Phi=0]} v \frac{\vartheta}{|\nabla \Phi|} \, \mathrm{d}s$$

Outline

Test Setting

$\partial\mathcal{O}$ is divided into 3 parts:

- Γ_D: the fixed Dirichlet boundary
- Γ_N: part of the Neumann boundary where the surface loads act on; this is also fixed and does not move during the optimization process
- Γ_0: all other parts of the boundary; this is the only part of $\partial\mathcal{O}$ to be optimized

Test Setting

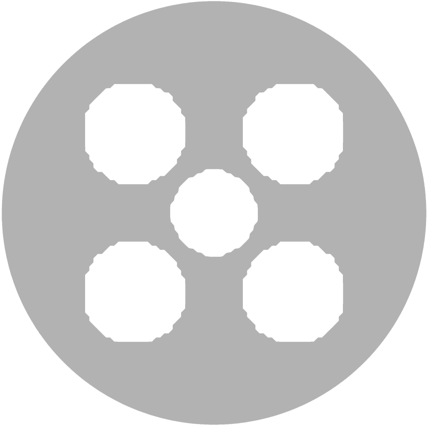
objective function (compliance with $f \equiv 0$):

$$J(\mathcal{O}, \omega) = \int_{\Gamma_N} g(\omega) \cdot u \, \mathrm{d}s + \ell R_i(\mathcal{O})$$

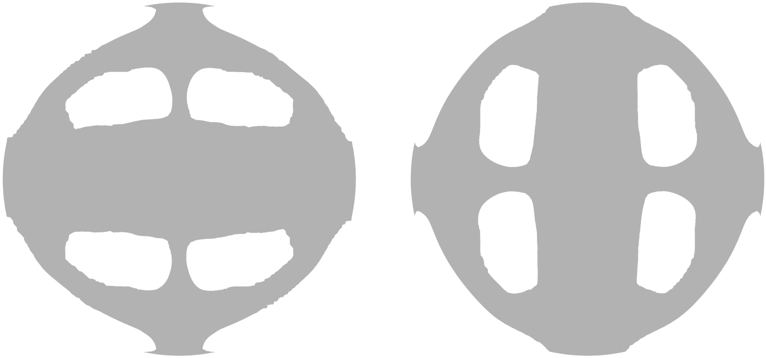
with regularization terms

$$\begin{aligned} R_1(\mathcal{O}) &= \int_{\partial\mathcal{O}} \mathrm{d}s \text{ (and volume preservation)} , \\ R_2(\mathcal{O}) &= \int_{\mathcal{O}} \mathrm{d}x \end{aligned}$$

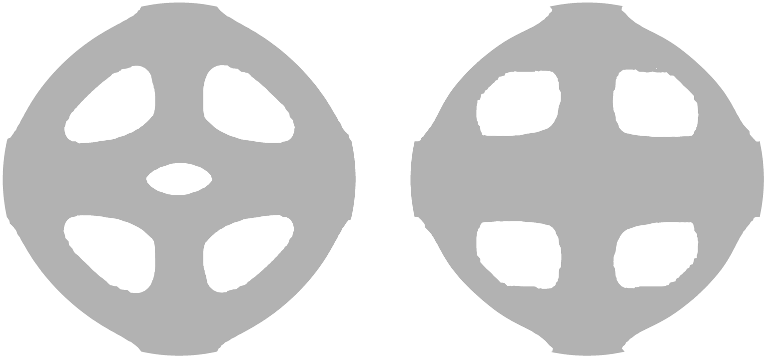
Instance 1 - Initial Shape



Instance 1 - Optimal shapes for g_0 and g_1



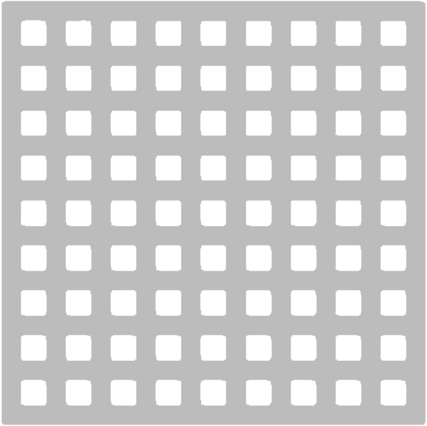
Instance 1 - Optimal shapes for $\frac{1}{2}g_0 + \frac{1}{2}g_1$ and 2 scenarios



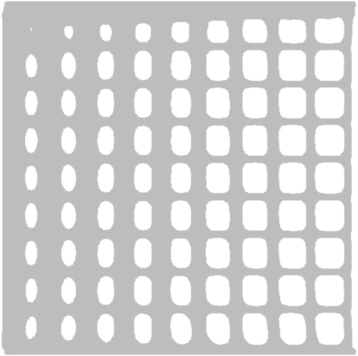
Instance 2 - Initial shape and optimal shape for g_0

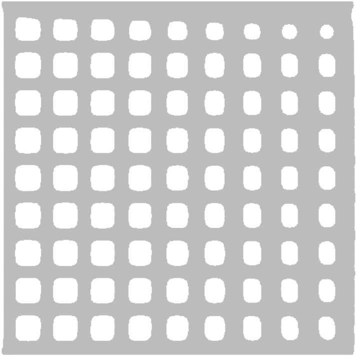
Instance 2 - Optimal shapes for $\frac{1}{2}g_0 + \frac{1}{2}g_1$ and 2 scenarios

Instance 3 - Initial shape

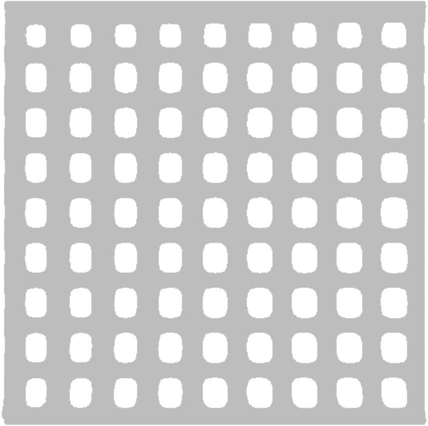


Instance 3 - Optimal shapes for g_0 and g_1

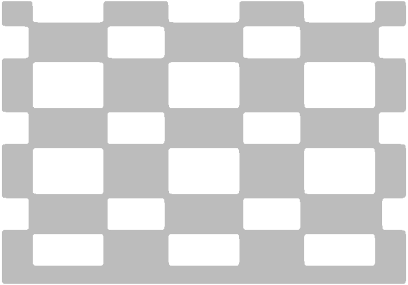




Instance 3 - Optimal shape 2 scenarios



Instance 4 - Initial and Optimal Shapes



Instance 5 - Optimal shape for g_0 and g_1 and 2 scenarios