REMARK ON THE REALIZATION OF COHOMOLOGY GROUPS

By C.-F. BÖDIGHEIMER

[Received 16th February 1981; in revised form 23rd November 1981]

Introduction

IT is well-known how to realize any abelian group G as the singular homology $H_n(X)$ of a space X, but in the case of cohomology there are certain restrictions on G to be realizable [**2**], [**5**] and [**8**]. However, the situation is more favourable with a Čech-type cohomology theory. Let H be a contravariant homotopy functor from compact spaces to abelian groups which is half-exact, reduced and commutes with inverse limits; assume further that for some $m \geq 1 H(S^{m+1})$ is the group $\mathbb{Z}$ of integers and $H(S^m)$ is torsion-free.

THEOREM. *For any abelian group G there exists a compact space X such that* $H(X) \cong G$.

1. The Construction

1.1. We choose a free presentation

$$0 \to F_1 \to F_0 \to G \to 0$$

of G and bases A in F_1, B in F_0. A is considered to be well-ordered, i.e. identified with the segment Ω_λ of all ordinals α less than a certain ordinal λ. Write $F = (f_{ab} \mid a \in A,\ b \in B)$ for the $A \times B$ integer matrix determined by the inclusion of F_1 in F_0.

As a first attempt we might realize F_0, F_1 by strong wedges of spheres $\bigvee_A S^{m+1}$, $\bigvee_B S^{m+1}$ (i.e. the inverse limits of all finite subwedges) since by the continuity of H we have

$$H\Big(\bigvee_A S^{m+1}\Big) \cong \bigoplus_A H(S^{m+1}) \cong F_1 \quad \text{and} \quad H\Big(\bigvee_B S^{m+1}\Big) \cong \bigoplus_B H(S^{m+1}) \cong F_0.$$

We should like to map $\bigvee_B S^{m+1}$ to $\bigvee_A S^{m+1}$ so as to realize the transposed matrix F. But a map f from S^{m+1} to $\bigvee_A S^{m+1}$ has its image contained in a countable subwedge of $\bigvee_A S^{m+1}$ (only countably many of the different counterimages $f^{-1}(S_a^{m+1} \setminus *)$ for $a \in A$ can be non-empty, since they are open and disjoint). This means that F cannot be realized if for some $b \in B$ there are uncountably many $a \in A$ with $f_{ab} \neq 0$. We get around this difficulty by constructing "long" spheres S_λ to replace the ordinary spheres S^{m+1} in $\bigvee_B S^{m+1}$.

1.2. Recall that for any ordinal λ there is a "long line" L_λ defined as $\Omega_\lambda \times [0, 1[$ with the order topology induced by lexicographic order. If λ is countable L_λ is the real half line; if λ is the first uncountable ordinal L_λ is the well-known Alexandroff line **[7]**, **[6]**. In any case it is a Hausdorff 1-manifold, not necessarily paracompact.

We shall regard Ω_λ as a subspace of L_λ by identifying α with $(\alpha, 0)$. Using the Alexander subbase theorem it is easy to see that the interval $[0, \lambda]$ in $L_{\lambda+1}$ is a compact space, hence the open subspace $L_\lambda = [0, \lambda[$ is locally compact. Let S_λ denote the one-point-compactification of $L_\lambda \setminus \{0\} =]0, \lambda[$, or in other words identify the end points 0 and λ of $[0, \lambda]$ to be the basepoint of S_λ.

1.3. The inclusions $L_\alpha \subset L_\lambda$ for $\alpha \leq \lambda$ induce maps $\Sigma_{\alpha\lambda}: S_\lambda \to S_\alpha$ smashing the subspace $[\alpha, \lambda]$ to the basepoint. Note that $\Sigma_{\beta\alpha}\Sigma_{\alpha\lambda} = \Sigma_{\beta\lambda}$ for $\beta \leq \alpha \leq \lambda$.

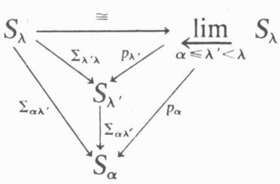
LEMMA. *For λ a limit ordinal the map $S_\lambda \to \varprojlim_{\alpha<\lambda} S_\alpha$ is a homeomorphism.*

Proof. For each point $x = (x_\alpha \mid \alpha < \lambda)$ of $\varprojlim_{\alpha<\lambda} S_\alpha$ there exists a certain ordinal $\alpha_0 < \lambda$ and some $t \in [0, 1[$ such that (i) x_α is the basepoint for $\alpha < \alpha_0$, and (ii) x_α equals (α_0, t) for $\alpha \geq \alpha_0$. Then $(\alpha_0, t) \in S_\lambda$ is the only point mapped onto x. Since both spaces are compact the map is homeomorphic.

1.4. For the functor H there is no difference between the spaces S_λ and ordinary spheres.

LEMMA. *For $1 \leq \alpha < \lambda$ the map $\Sigma_{\alpha\lambda}$ induces an isomorphism $H(S_\lambda) \to H(S_\alpha)$.*

Proof. We prove the lemma by transfinite induction on λ. The lemma is obvious for $\lambda = 1$. If it is true for λ, so it is for $\lambda + 1$, since $\Sigma_{\lambda,\lambda+1}$ is a homotopy equivalence. For λ a limit ordinal and a fixed $\alpha \leq \lambda$ assume the statement to be true for all λ' with $1 \leq \alpha \leq \lambda' < \lambda$. In the diagram

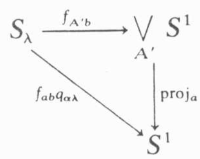


all maps $\Sigma_{\alpha\lambda'}$ in the inverse system induce isomorphism by hypothesis, whence the same is true for the projections $p_{\lambda'}$. It follows that $\Sigma_{\lambda'\lambda}$ and especially $\Sigma_{\alpha\lambda}$ induce isomorphism.

We note $H(S_\lambda) \cong H(S^1)$, and in the same way we have $H(S^m S_\lambda) \cong H(S^{m+1}) \cong \mathbb{Z}$ and $H(S^{m+1} S_\lambda) \cong H(S^m)$.

1.5. Another family of maps $q_{\alpha\lambda}\colon S_\lambda \to S^1$ for $\alpha < \lambda$ identifies $[0, \alpha] \cup [\alpha+1, \lambda]$ to a point. For $\lambda = \alpha + 1$ not a limit ordinal identifying $[0, \alpha]$ to a point is homotopic to identifying $[1, \alpha+1]$ to a point. Hence $q_{\alpha\lambda} = q_{\alpha,\alpha+1}\Sigma_{\alpha+1,\lambda} \simeq \Sigma_{1,\alpha+1}\Sigma_{\alpha+1,\lambda} = \Sigma_{1,\lambda}$. The pull-back $(S^m q_{\alpha\lambda})^*(e)$ of a generator $e \in H(S^{m+1}) \cong \mathbb{Z}$ is therefore a generator of $H(S^m S_\lambda)$ and independent of α. It will also be denoted by e.

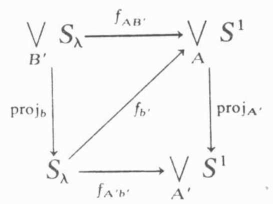
1.6. Remember that A was identified with Ω_λ and let a correspond to α in our notation. Now pick some fixed $b \in B$. A map $f_{ab}q_{\alpha\lambda}\colon S_\lambda \to S^1 \to S^1$ sends the complement of $]\alpha, \alpha+1[$ to the basepoint; for a finite subset A' of A the finitely many of them with $a \in A'$ can therefore easily be combined to a map $f_{A'b}$ such that



commutes for each $a \in A'$.

All these maps $f_{A'b}$ for A' finite in A form a map into the inverse system of all finite wedges $\bigvee_{A'} S^1$; so we have a map $f_b\colon S_\lambda \to \bigvee_A S^1$ such that $\mathrm{proj}_a f_b = f_{ab}q_{\alpha\lambda}$. The advantage of S_λ over the sphere S^1 lies in these maps f_b.

1.7. For A' finite in A the set of all $b' \in B$ with $f_{a'b'} \neq 0$ for some $a' \in A'$ is a finite subset B'. The wedge sum of all $f_{b'}$ for $b' \in B'$ yields a map $f_{AB'}$ fitting into the following commutative diagram.



By passing to limits with the maps $f_{A'B'} = \mathrm{proj}_{A'} f_{AB'}$ we finally get a map f

in a commutative diagram

$$\begin{array}{ccc}
\bigvee_B S_\lambda & \xrightarrow{f} & \bigvee_A S^1 \\
\uparrow{\scriptstyle \mathrm{incl}_b} & & \downarrow{\scriptstyle \mathrm{proj}_a} \\
S_\lambda & \xrightarrow[f_{ab}q_{\alpha\lambda}]{} & S^1
\end{array}$$

where the bottom map induces multiplication by f_{ab}, i.e. sends $e \in H(S^{m+1})$ to $f_{ab} \cdot e \in H(S^m S_\lambda)$.

2. The Proof

Let X denote the mapping cone of $S^{m-1}f$. In the cofibration sequence

$$\begin{array}{ccccccccc}
H\left(\bigvee_B S^{m-1}S_\lambda\right) & \xleftarrow{(S^{m-1}f)^*} & H\left(\bigvee_A S^m\right) & \leftarrow H(X) \leftarrow & H\left(\bigvee_B S^m S_\lambda\right) & \xleftarrow{(S^m f)^*} & H\left(\bigvee_A S^{m+1}\right) \\
\wr\Vert & & \wr\Vert & & \wr\Vert & & \wr\Vert \\
F_0 \otimes H(S^m) & \xleftarrow[F\otimes 1]{} & F_1 \otimes H(S^m) & & F_0 \otimes H(S^{m+1}) & \xleftarrow[F\otimes 1]{} & F_1 \otimes H(S^{m+1})
\end{array}$$

$S^m f$ and $S^{m-1}f$ induce the matrix F and we end up with a short exact sequence

$$0 \to G \otimes H(S^{m+1}) \to H(X) \to \operatorname{Tor}(G, H(S^m)) \to 0.$$

Our assumptions on $H(S^{m+1})$ and $H(S^m)$ imply the result.

3. Some Applications

3.1. The best-known continuous cohomology theories are Čech cohomology and complex K-theory. Given any abelian group C_n $(n \geq 2)$ we take $m = n-1$ for $\check{H}^n$ and construct spaces X_n such that $\check{H}^n(X_n) \cong C_n$. Because we know in addition that $\check{H}^i(X_n) = 0$ if $i \neq n$, we have simultaneously $\check{H}^n(X) \cong C_n$ for $X = \bigvee X_n$. In complex K-theory we construct in the same manner for any two abelian groups A, B a compact space X such that $\tilde{K}^0(X) \cong A$ and $\tilde{K}^1(X) \cong B$.

3.2. For K-homology there is the split exact universal coefficient sequence of **[1]** for compact metric spaces

$$0 \to \operatorname{Ext}(\tilde{K}^{n+1}(X), \mathbb{Z}) \to \tilde{K}_n(X) \to \operatorname{Hom}(\tilde{K}^n(X), \mathbb{Z}) \to 0. \quad (3.2)$$

According to **[2]**, **[3]** and **[5]**, **[8]** this implies conditions on the structure of $\tilde{K}_n(X)$. On the other hand it implies realizability results: as our construction produces metric spaces for countable groups, any group of the form $\operatorname{Ext}(B, \mathbb{Z}) \otimes \operatorname{Hom}(A, \mathbb{Z})$ with A, B countable occurs as $\tilde{K}_n(X)$ for some compact metric X.

3.3. For CW complexes X there is a dual split exact universal coefficient sequence in [**4**]

$$0 \to \operatorname{Ext}(\tilde{K}_{n-1}(X), \mathbb{Z}) \to \tilde{K}^n(X) \to \operatorname{Hom}(\tilde{K}_n(X), \mathbb{Z}) \to 0. \quad (3.3)$$

As in 3.2 it follows that in the category of CW complexes X we do have conditions on the structure of $\tilde{K}^n(X)$. Therefore our theorem cannot hold if we require the realizing space X to be a CW complex. Vice versa we see that the sequence (3.3) cannot split for arbitrary compact X, since we have learned that in the category of compact spaces there are no restrictions for $\tilde{K}^n(X)$.

3.4. There is a counterexample even to the existence of the sequence (3.3) for compact metric spaces. For the solenoid $X = \varprojlim_k (\cdots S^1 \overset{k}{\leftarrow} S^1 \cdots)$ we find $\tilde{K}^1(X) = \mathbb{Q}$ and $\tilde{K}^0(X) = 0$. Using (3.2) we note $\tilde{K}_0(X) = \operatorname{Ext}(\mathbb{Q}, \mathbb{Z}) = \mathbb{R}$ and $\tilde{K}_1(X) = \operatorname{Hom}(\mathbb{Q}, \mathbb{Z}) = 0$. Substitution in (3.3) would imply the contradiction $\tilde{K}^1(X) = \operatorname{Ext}(\mathbb{R}, \mathbb{Z})$ which is a countable product of the reals.

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Mathematisches Institut
Im Neuenheimer Feld 288
D-6900 Heidelberg
West Germany