ON THE TOPOLOGY OF
MODULI SPACES OF
RIEMANN SURFACES

Part I

Hilbert Uniformization

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"Übrigens ist zu bemerken, dass im Falle nicht-schlichtartiger Bereiche Ω die Schlitze in gewisser Weise einander zugeordnet werden, sodass die Zusammenhangsverhältnisse des Schlitzbereichs vermöge dieser Zuordnung mit denen von Ω übereinstimmen."

D. Hilbert,
"Ausgewählte Kapitel der Funktionentheorie"

Preface

On a Riemann surface assume a point and a direction at this point is given, and imagine an electrical dipol pointing in the given direction. The resulting flow has several stagnation points connected by the critical stream lines. If the surface is dissected along these critical stream lines, the flow has a complex potential, which maps the complement of critical stream lines conformally onto a domain in the complex plane, whose complement consists of pairs of infinite, horizontal slits. Up to some normalization, the configuration and pairing of the slits is a conformal invariant of the directed surface.

This function, which associates to the conformal equivalence class of a closed, directed Riemann surface of fixed genus the similarity class of a configuration of slit pairs is proved to be a homotopy equivalence between the moduli space of directed Riemann surfaces and this configuration space, called the space of parallel slit domains. At the same time this moduli space is homotopy equivalent to the classifying space of the mapping class group of surfaces with one boundary curve.

This new description of the moduli space is based on old ideas of geometric function theory. But it turns out to be useful for studying the homology of the moduli space and of the mapping class group. These applications, sketched below, will be given in subsequent parts.

Introduction

We consider closed Riemann surfaces F of arbitrary genus g, together with a basepoint P, and a direction x at P.

The isotropy classes of orientation-preserving diffeomorphisms $\gamma : F \longrightarrow F$ keeping the direction x fixed is called the directed mapping class group $\vec{\Gamma}(g) = \Gamma(F,x)$. It is an extension of the based mapping class group $\Gamma^{\cdot}(g) = \Gamma(F,P)$, which itself is an extension of the (free) mapping class group $\Gamma(g) = \Gamma(F)$,

$$0 \longrightarrow \mathbf{Z} \longrightarrow \vec{\Gamma}(g) \longrightarrow \Gamma^{\cdot}(g) \longrightarrow 1 \quad ,$$

$$1 \longrightarrow \pi_1 F \longrightarrow \Gamma^{\cdot}(g) \longrightarrow \Gamma(g) \longrightarrow 1 \quad .$$

If C is a nullhomotopic curve through P, let $\Gamma(g,1) = \Gamma(F,C)$ be the group of isotopy classes of diffeomorphisms keeping C pointwise fixed. As a first result we mention

<u>Theorem I.</u> $\qquad \Gamma(g,1) \cong \vec{\Gamma}(g)$.

Two directed Riemann surfaces (F_1,x_1) and (F_2,x_2) are called conformally equivalent if there is a conformal map $F_1 \longrightarrow F_2$ sending x_1 to x_2. The set $\vec{\mathfrak{M}}(g)$ of conformal equivalence classes $[F,x]$, with the Teichmüller metric of maximal dilatation of quasiconformal maps, is the moduli space of directed Riemann surfaces. There are forgetful maps onto the based and (free) moduli space $\vec{\mathfrak{M}}(g) \longrightarrow \mathfrak{M}^{\cdot}(g) \longrightarrow \mathfrak{M}(g)$. Like the classical situations there is a contractible Teichmüller space $\vec{\mathfrak{T}}(g) \cong \mathbf{R}^{6g-3}$, on which $\vec{\Gamma}(g)$ acts properly discontinuously, and $\vec{\mathfrak{M}}(g)$ is the orbit space. But unlike the classical situation, the action of $\vec{\Gamma}(g)$ on $\vec{\mathfrak{T}}(g)$ is free, simply because an automorphism of a Riemann surface holding a direction fixed must be the identity. Therefore $\vec{\mathfrak{M}}(g)$ has the homotopy type of the classifying space of $\vec{\Gamma}(g)$,

<u>Theorem II.</u> $B\vec{\Gamma}(g) \simeq \vec{\mathfrak{M}}(g)$.

Now for the uniformization, there is (up to normalization) a unique harmonic function $u : F \longrightarrow \bar{R}$ with a dipol at P in the direction X . Let $\mathfrak{K}_o$ be the critical graph of the gradient flow Φ_u of u , i.e. all integral curves connecting stagnation points S_i . On its simply-connected complement $F_o = F - \mathfrak{K}_o$ there is a unique harmonic function v conjugate to u . As already indicated, the Hilbert uniformization uses the geometry of Φ_u and combinatoric of $\mathfrak{K}_o$ as moduli for the conformal structure of F . The continuous part of the moduli data consists of the positions of the stagnation points, i.e. the values of u and the boundary values of v at all S_i . The discrete part of the moduli is more intricate: as there are multiple stagnation points with Morse index $m > 1$ for a non-generic surface, there are fewer critical curves than there should be. By adding "virtual" critical curves we obtain the branching complex $\bar{F} \cup \mathfrak{B}$ with the properties : (1) $\bar{F} \cup \mathfrak{B}$ has $4g$ double-edges B_i (being right and left branches of $2g$ virtual critical curves), (2) there is a pairing of these B_i , (3) u and v extend to $\bar{F} \cup \mathfrak{B}$ (4) the B_i are linearly ordered by the (constant) value of v . The complex potential $w = u + iv$ maps the branching curves B_i , onto $4g$ infinite slits in $\mathbb{C}$, parallel to the x-axis.

This makes us introduce the space $PS\mathbb{C}(g)$ of parallel slit domains in $\mathbb{C}$ of genus g . An element $\mathfrak{L}$ is represented by a configuration $L = (L_1,\dots,L_{4g};\lambda)$ of horizontal, infinite slits L_i , ordered by a permutation $\lambda \in \Sigma_{4g}$. L is subject to a non-degeneracy condition. There is an equivalence relation, solely responsible for the homological structure of $PS\mathbb{C}(g)$. There is a metric, and $PS\mathbb{C}(g)$ is a connected cell complex of dimension $6g$.

domains are introduced in chapter 4 . We restrict to give the basic properties of the space $PS\mathbb{C}(g)$. (IV) is noted under (4.9.8) . Finally, chapter 5 brings together chapter 2 and the preparations of chapters 3 and 4 to prove (III) as Theorem (5.5.1) .

Further Aspects.

For many purposes of surface theory this uniformization might be awkward. We find it therefore appropriate - and necessary to justify this work and the reader's patience - to hint at several applications, generalizations and speculations.

(1) What $PS\mathbb{C}(g)$ as a model for $B\Gamma(g,1)$ or $\vec{\mathfrak{M}}(g)$ has immediately to offer is a cell decomposition and stratification. It is similar to decompositions of other configuration spaces such as the classifying spaces of symmetric groups and braid groups, and might be useful for homological computations. We will describe this cell structure in Part IV.

(2) There is an obvious "adding-a-new-handle" map $\sigma : PS\mathbb{C}(g) \longrightarrow PS\mathbb{C}(g+1)$. Configuration space models have been used to show the homological stability of similar maps, e.g. for symmetric and braid groups. So far σ is known to induce an isomorphism in 1/8 of the homological range of $PS\mathbb{C}(g)$, cf. [Harer 1985], [Ivanov 1987].

(3) There is an addition map $\mu : PS\mathbb{C}(g_1) \times PS\mathbb{C}(g_2) \longrightarrow PS\mathbb{C}(g_1+g_2)$ by putting one configuration in the upper, the other in the lower half-plane. This induces a multiplication in homology, see [Miller 1986]. More so, by varying the patches into which the two (or several) configurations are implanted in the plane, one gets parametrized families of such additions; they induce homology operations indexed by homology classes of braid groups.

We will do this in Part II, and will use it in Part III to find new classes in the homology of the mapping class group.

(4) By the geometry of the model $PS\mathbb{C}(g)$ one can define interesting maps into $PS\mathbb{C}(g)$. For example, letting some slit pairs vary along fixed trails in $\mathbb{C}$, this "position manifold" defines an immersed surface in some $PS\mathbb{C}(g)$, carrying the generator of $H_2\Gamma(g,1)$, [Harer 1983].

(5) We have applied this uniformization procedure to closed surfaces. It works also for surfaces with boundary and punctures; each boundary circl gives a finite slit, and each puncture a distinguished point.

(6) Teichmüller and moduli spaces possess several important compactification: Looking at the regularity conditions we see ways to compactify $PS\mathbb{C}(g)$ by admitting some kinds of degenerated configurations. For example, admit ting subconfigurations as in (4.4.10) and non-connected pairing function: (4.4.6) gives spaces as in [Bers 1974, 1975], [Abikoff 1977].

(7) As a final example, we point out a connection to ergodic theory. The gradient flow of a dipol function has the Poincare (first return) functic It is the map from the right to the left equipotential line of the suppor see (4.4.5), (4.8.6) . This is called an interval exchange map $V^+ \longrightarrow V^-$ see [Keane 1975], [Veech 1978]. The connection to surface theory via measured foliations is well-known, cf. [Strebel 1984, p.58]. First investigations of a "space of interval exchange transformations $\mathfrak{I}(g)$ in connection with moduli spaces are [Veech 1982], [Mazur 1982]. But there one associates to a "weighted" interval exchange transformation a Riemanr surface. The space $PS\mathbb{C}(g)$ offers a natural map in the other direction $PS\mathbb{C}(g) \longrightarrow \mathfrak{I}(g)$. This map is surjective and seems to have nearly contractible fibres; furthermore, the dimension $4g-2$ of $\mathfrak{I}(g)$ is

precisely the homological dimension of $\Gamma(g,1)$, see [Harer 1986]. It should not be a mere curiosity that the equivalence classes for parallel slit domains amounts for the pairing function to Ranzy classes of permutations, see [Ranzy 1979].

History.

The history of this uniformization method begins - after the heuristic period, cf. [Klein 1982] - with a talk of Hilbert in the Mathematische Gesellschaft in Göttingen in April 1909 during a visit of Poincare. This talk, published as [Hilbert 1909], proves the existence of dipol functions using the resurrected Dirichlit's principle from [Hilbert 1904]. During the summer term of 1909 Hilbert lectured on "Ausgewählte Kapitel der Funktionentheorie (Konforme Abbildungen)"; of this lecture there exist unpublished notes by his student R. Courant, [Hilbert 1909b]. In these notes one finds only the remark which we have chosen as a motto, but in [Hilbert 1909] the parallel slit domains occur explicitly with figures. Hilbert's proof became standard in all textbooks, starting with [Weyl 1913], [Hurwitz-Courant 1929], for the case $g = 0$; here this uniformization method is well-known, especially after Koebe's work [1909, 1910], and Courant's [1912a, 1912b], see e.g. [Ahlfors 1953, p.259-261], [Cohn 1967, p.196-200], [Nehari 1952, chap.VII]. In the case of a multiply-connected schlicht surface the connection to configuration spaces of points and slits in the plane is obvious, cf. [Bers 1960] and [Jenkins 1957], where the Teichmüller metric on the moduli space and the euclidean metric on the configuration space are compared.

But apart from [Courant 1919, 1941, 1950] and [Koebe 1919] the parallel slit domains for higher genera occur in the field of minimal surfaces, e.g.

[Shiffman 1939], or [Luckhaus 1978], and earn criticism [Jost 1985, introd.].

Our interest arose from reading [Giddings-Martinec-Witten 1986] and [Giddings-Wolpert 1987]. As a morphism in 1-dimensional conformal field theory Riemann surfaces with incoming and outgoing boundary circles are discussed using parallel slit domains.

We mention [Saito 1987] and [Arbarello-De Concini-Kac-Procesi 1988] where the moduli space $\vec{\mathfrak{M}}(g)$ also occurs.

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Chapter 1

Diffeomorphism Groups and Mapping Class Groups

In this first chapter we introduce various diffeomorphism groups and mapping class groups. There are three kinds of diffeomorphisms: arbitrary ones, those keeping a given point fixed, and those keeping a given tangential vector fixed (up to a positive stretching factor). It is particularly important, that the groups of these diffeomorphisms have contractible identity components, if the genus of the surface is at least 2 .

From these diffeomorphism groups we get the usual, the pointed and the directed mapping class group. The second and the third are extensions of the former. To connect our treatment to the literature, we will identify the directed mapping class group with mapping class group of a surface with one boundary circle.

1.1 Direction bundles.

Let F denote a connected, compact and oriented surface without boundary. The genus of F will be g. Furthermore, F will be smooth, i.e. F has an atlas of neighbourhoods Z_α and local coordinates or local parameters z_α from Z_α onto open subsets W_α of $\mathbb{R}^2$ such that all transition functions are differentiable of class C^∞.

Denote the tangent bundle of F by $T(F) \longrightarrow F$. On the complement of its zero-section we introduce an equivalence relation: $x_1 \sim x_2$ if and only if $x_1 = ax_2$ for some positive real number a. The equivalence class $x = \vec{x}$ of a non-zero tangent vector x is called a <u>direction</u>, and the quotient space, called the direction bundle, is denoted by $\vec{T}(F)$. Any immersion $f : F_1 \longrightarrow F_2$ between surfaces induces a map $\vec{Df} : \vec{T}(F_1) \to \vec{T}(F_2)$ of direction bundles by $\vec{Df}(\vec{x}) = \overrightarrow{Df(x)}$, where Df is the differential of f. This S^1-bundle is diffeomorphic to the unit tangent bundle of F, and a closed 3-manifold. Furthermore, for $g \geq 1$, it is an Eilenberg-MacLane space $K(G,1)$ for $G = \pi_1\vec{T}(F)$, which fits into the exact sequence

$$(1.1.1) \qquad 1 \longrightarrow \mathbb{Z} \longrightarrow \pi_1\vec{T}(F) \longrightarrow \pi_1 F \longrightarrow 1 .$$

This extension is central and classified by the cohomology class in $H^2(\pi_1(F);\mathbb{Z})$ which corresponds to the Euler class of the tangent bundle under $H^2(B\pi_1(F)) \cong H^2(F)$.

There will be a point P on F chosen, and in addition a direction $x \in \vec{T}_P(F)$ at P. Let $D \subseteq F$ be an open disc centered at P, with a smooth parametrization by the unit disc $h : \mathbb{D} \longrightarrow D$ such that $h(0) = P$, and the direction of dx corresponds to x under the differential of h, i. e. $\vec{Dh}(\overrightarrow{dx}) = x$. Later we shall need a surface with one boundary component; we take $F^o = F-D$.

Throughout the next chapters we will always refer to this same surface F; the point P, and the direction x.

1.2 Mapping class groups of a closed surface.

Since all surfaces we will consider are oriented, all diffeomorphisms, homeomorphisms, etc. between surfaces shall be orientation-preserving (unless there is an explicit statement to the contrary).

Let Diff(F) denote the group of all smooth diffeomorphisms $\gamma : F \to F$. With the C^∞-topology (of uniform convergence on compact subsets of differential of all orders) this is an infinite-dimensional, topological group, modeled on inverse limits of smooth Hilbert manifolds; in particular, it is a Frechet manifold, and therefore metrizable, absolute neighbourhood retracts; see [Earle-Eells 1969], [Fischer-Tromba 1984a], [Omori 1970]. The subgroup $\mathrm{Diff}_o(F)$ of diffeomorphisms isotopic (equivalently, homotopic) to the identity is a closed, normal subgroup; it is the path component of the identity. In our treatment this is now the convenient place to quote the following fundamental result about $\mathrm{Diff}_o(F)$.

(1.2.1) PROPOSITION. $\mathrm{Diff}_o(F)$ is contractible for $g \geq 2$. ■

But the proof belongs to the main parts of Teichmüller-theory, which we will sketch in chapter 2. In fact, the contractibility of $\mathrm{Diff}_o(F)$ is equivalent to the contractability of the Teichmüller space, see (2.2), (2.4). The first proof is in [Earle-Eells 1969; p. 34] and uses Teichmüller's theorem; but they mention (p. 35) an independent proof which relys on work of H. Lewy, E. Heinz and J. H. Sampson, and incomplete proofs of K. Shibata; see [Jost 1984, p. 106] for a discussion. Meanwhile there are complete proofs purely in the framework of differential geometry, see [Fischer-Tromba 1984 a,b,c, 1987]. For an interesting approach using only the "Teichmüller theory of diffeomorphisms of the circle", see [Earle-McMullen 1986], which is based on [Donady-Earle 1986].

The mapping class group of F is defined as the group of homotopy classes of diffeomorphisms,

$$(1.2.2) \qquad \Gamma(F) := \pi_0 \mathrm{Diff}(F) = \mathrm{Diff}(F)/\mathrm{Diff}_0(F) \ .$$

This group, introduced and studied by J. Nielsen and M. Dehn, see [Nielsen 1927], [Dehn 1938], is isomorphic to $\mathrm{Out}(\pi_1 F) \cong \mathrm{Aut}(\pi_1 F)/\mathrm{Inn}(\pi_1 F)$, see also [Mangler 1939], or [Magnus-Karrass-Solitar 1966, p. 176].

For example, $\Gamma(F)$ is trivial for F the sphere, and $\Gamma(F) \cong SL_2(\mathbb{Z})$ for F the torus. For $g \geqq 1$, $\Gamma(F)$ is infinite, and finitely presentable, [Dehn 1938], [Lickorish 1964], [Hatcher-Thurston 1980]; see [Birman 1974].

For $g \geqq 2$ we conclude from (1.2.1) for the classifying spaces

$$(1.2.3) \qquad B\Gamma(F) \simeq B\mathrm{Diff}(F) \ .$$

We are interested in two extensions of $\Gamma(F)$. To define the first recall the basepoint $P \in F$. Denote by $\mathrm{Diff}(F,P)$ the group of all diffeomorphisms $\gamma : F \longrightarrow F$ fixing P ; and denote by $\mathrm{Diff}_0(F,P)$ the subgroup of those γ which are homotopic to the identity by a homotopy keeping P fixed throughout. Note that $\mathrm{Diff}_0(F,P)$ is the path component of the identity in $\mathrm{Diff}(F,P)$ as well as in $\mathrm{Diff}(F,P) \cap \mathrm{Diff}_0(F)$. Obviously, $\mathrm{Diff}(F,P)$ and $\mathrm{Diff}_0(F,P)$ are closed subgroups in $\mathrm{Diff}(F)$, and $\mathrm{Diff}_0(F,P)$ is normal in $\mathrm{Diff}(F,P)$.

The evalutation map $\varepsilon : \mathrm{Diff}(F) \longrightarrow F$, $\varepsilon(\gamma) = \gamma(P)$ and its restriction ε_0 to $\mathrm{Diff}_0(F)$ induce two fibrations in the commutative diagram

$$(1.2.4) \qquad \begin{array}{ccccccc}
\Omega F & \longrightarrow & \mathrm{Diff}(F,P) & \longrightarrow & \mathrm{Diff}(F) & \xrightarrow{\varepsilon} & F \\
\| & & \uparrow & & \uparrow & & \| \\
\Omega F & \longrightarrow & \mathrm{Diff}(F,P) \cap \mathrm{Diff}_0(F) & \longrightarrow & \mathrm{Diff}_0(F) & \xrightarrow[\varepsilon_0]{} & F \\
\uparrow & & \uparrow & & & & \\
\Omega_0 F & \longrightarrow & \mathrm{Diff}_0(F,P) & & & &
\end{array}$$

Since $\mathrm{Diff}_o(F)$ is contractible for $g \geqq 2$, we have $\mathrm{Diff}_o(F,P) \simeq \Omega_o F$, the component of null-homotopic loops on F , which proves the following

(1.2.5) PROPOSITION. $\mathrm{Diff}_o(F,P)$ *is contractible for* $g \geqq 2$. ■

We define the pointed mapping class group as the group of pointed homotopy classes of pointed diffeomorphisms,

(1.2.6) $\Gamma(F,P) := \pi_o(\mathrm{Diff}\ (F,P)) = \mathrm{Diff}(F,P)/\mathrm{Diff}_o(F,P)$.

If $g \geqq 2$, the upper fibration of (1.2.4) and (1.2.1) yield the exact sequence

(1.2.7) $1 \longrightarrow \pi_1 F \longrightarrow \Gamma(F,P) \longrightarrow \Gamma(F) \longrightarrow 1$.

The two exceptional cases are: $\Gamma(F,P) = \Gamma(F) = 1$ for $g = 0$, and $\Gamma(F,P) = \Gamma(F) = SL_2(\mathbb{Z})$ for $g = 1$. Another consequence of (1.2.5) is

(1.2.8) $B\Gamma(F,P) \simeq B\mathrm{Diff}(F,P)$.

Note that $\Gamma(F,P)$ is also the mapping class group of the punctured surface $F_o = F-P$; cf. [Birman 1974, p. 148].

For the second extension of $\Gamma(F)$ recall the direction x . Let $\mathrm{Diff}(F,x)$ be the group of diffeomorphisms $\gamma : F \longrightarrow F$ fixing x , i.e. $\gamma(P) = P$ and $\vec{D}\gamma(x) = x$. And $\mathrm{Diff}_o(F,x)$ is the subgroup of all those γ homotopic to the identity by a homotopy keeping x fixed throughout. $\mathrm{Diff}_o(F,x)$ is the component of the identity in $\mathrm{Diff}(F,x)$ and in $\mathrm{Diff}(F,x) \cap \mathrm{Diff}_o(F)$, and is closed and normal in $\mathrm{Diff}(F,x)$.

As above, consider the diagram of evaluation fibrations $\vec{\varepsilon} : \mathrm{Diff}(F) \rightarrow \vec{T}(F)$, $\vec{\varepsilon}(\gamma) = \vec{D}\gamma(x)$. $\vec{\varepsilon}_o$ is the restriction to $\mathrm{Diff}_o(F)$.

(1.2.9)
$$\begin{array}{ccccccc}
\Omega\vec{T}(F) & \longrightarrow & \mathrm{Diff}(F,x) & \longrightarrow & \mathrm{Diff}(F) & \xrightarrow{\vec{\varepsilon}} & \vec{T}(F) \\
\| & & \uparrow & & \uparrow & & \| \\
\Omega\vec{T}(F) & \longrightarrow & \mathrm{Diff}(F,x) \cap \mathrm{Diff}_o(F) & \longrightarrow & \mathrm{Diff}_o(F) & \xrightarrow[\vec{\varepsilon}_o]{} & \vec{T}(F) \\
\uparrow & & \uparrow & & & & \\
\Omega_o\vec{T}(F) & \longrightarrow & \mathrm{Diff}_o(F,x) & & & &
\end{array}$$

Using the contractibility of $\mathrm{Diff}_o(F)$ for $g \geqq 2$, and of $\Omega_o\vec{T}(F)$ for $g \geqq 1$, we conclude

(1.2.10) <u>PROPOSITION.</u> $\mathrm{Diff}_o(F,x)$ <u>is contractible for</u> $g \geqq 2$. ∎

The <u>directed mapping class group</u> is defined as

(1.2.11) $\Gamma(F,x) =: \pi_o\mathrm{Diff}(F,x) = \mathrm{Diff}(F,x)/\mathrm{Diff}_o(F,x)$.

Of course, $\mathrm{Diff}(F,x)$ is a subgroup of $\mathrm{Diff}(F,P)$, and the diagram

(1.2.12)
$$\begin{array}{ccccc}
\mathrm{Diff}(F,x) & \longrightarrow & \mathrm{Diff}(F,P) & \xrightarrow{\vec{\varepsilon}_P} & \vec{T}_P(F) \\
\| & & \downarrow & & \downarrow \\
\mathrm{Diff}(F,x) & \longrightarrow & \mathrm{Diff}(F) & \xrightarrow{\vec{\varepsilon}} & \vec{T}(F) \\
 & & \varepsilon\downarrow & & \downarrow \\
 & & F & = & F
\end{array}$$

where $\vec{\varepsilon}_P$ is the restriction of $\vec{\varepsilon}$, exhibits $\Gamma(F,x)$ for $g \geqq 2$ as an extension of $\Gamma(F)$ by $\pi_1\vec{T}(F)$ and of $\Gamma(F,P)$ by $\pi_1\vec{T}_P(F) \cong \mathbf{Z}$; it contains the exact sequences (1.1.1) and (1.2.7) :

(1.2.13)

$$
\begin{array}{ccccccccc}
 & & & & & & 1 & & \\
 & & & & & & \downarrow & & \\
 & & 1 & & & & \pi_1 F & & \\
 & & \downarrow & & & & \downarrow & & \\
1 & \longrightarrow & \mathbf{Z} & \longrightarrow & \Gamma(F,x) & \longrightarrow & \Gamma(F,P) & \longrightarrow & 1 \\
 & & \downarrow & & & & \downarrow & & \\
1 & \longrightarrow & \pi_1 \vec{T}(F) & \longrightarrow & \Gamma(F,x) & \longrightarrow & \Gamma(F) & \longrightarrow & 1 \\
 & & \downarrow & & & & \downarrow & & \\
 & & \pi_1(F) & & & & 1 & & \\
 & & \downarrow & & & & & & \\
 & & 1 & & & & & &
\end{array}
$$

Finally, note that

(1.2.14) $$B\Gamma(F,x) \simeq B\,\mathrm{Diff}(F,x) \, .$$

The importance of the equivalences (1.2.3), (1.2.8), and (1.2.14) lies in the fact that the diffeomorphism groups act on various spaces (e.g. F), whereas the existence of a corresponding action of the mapping class group amounts to a Nielsen realization problem.

A mapping class in $\pi_1(F) \subseteq \Gamma(F,P)$ is represented by the composition of two Dehn twists along parallel, disjoint curves, but having opposite directions; see figure in [Birman 1974; p. 159]. A mapping class in $\mathbf{Z} \subseteq \Gamma(F,x)$ is represented by a multiple of a full "spiral twist" of a disc around P; see figure (adjusted) in [Birman 1974; p. 167], and (2.3.17) .

1.3 Mapping class groups of a surface with boundary.

We will identify $\Gamma(F,x)$ with the mapping class group of the surface $F^o = F-D$. Recall that F^o has one boundary circle $C = \partial F^o$ parametrized by $h|S^1$. Let $Diff(F,D)$ be the group of all diffeomorphisms $\gamma : F \longrightarrow F$ which fix D pointwise; and let $Diff_o(F,D)$ denote the subgroup of all $\gamma \in Diff(F,D)$ homotopic to the identity by a homotopy keeping each point of D fixed throughout. $Diff(F,D)$ is a closed subgroup of $Diff(F)$; $Diff_o(F,D)$ is closed and normal in $Diff(F,D)$ and the component of the identity in $Diff(F,D)$ and in $Diff(F,D) \cap Diff_o(F)$. Of course, $Diff(F,D)$ can be regarded as $Diff(F^o, \partial F^o)$, the group of diffeomorphisms of F being the identity on the boundary ∂F^o.

There are the following inclusions

(1.3.1)
$$\begin{array}{ccc} Diff(F,D) & \longrightarrow & Diff(F,x) \\ \cup\!\shortmid & & \cup\!\shortmid \\ Diff_o(F,D) & \longrightarrow & Diff_o(F,x) \end{array}$$

(1.3.2) PROPOSITION. *The inclusion* $Diff(F,D) \longrightarrow Diff(F,x)$ *is a weak homotopy equivalence.*

For the proof we study the space $E_x(F)$ of all smooth embeddings $f : \mathbf{D} \longrightarrow F$ such that $f(0) = P$ and $Df(\vec{dx}) = \vec{x}$. Note that $h \in E_x(F)$. Composing $\gamma \in Diff(F,x)$ with the basepoint $h : \mathbf{D} \longrightarrow F$ of $E_x(F)$ defines a continuous map

(1.3.3)
$$\varepsilon_D : Diff(F,x) \longrightarrow E_x(F) , \quad \varepsilon_D(\gamma) = \gamma \cdot h$$

whose fibre over h is $Diff(F,D)$. A based version of [Hirsch; Theorem 3.1, p. 185] shows, that $E_x(F)$ is path-connected, and at the same time that ε_D is surjective. Furthermore, Diff(F) is locally contractible being a Hilbert space manifold; it follows that ε_D possesses local sections; thus ε_D is a locally trivial fibre bundle with fibre Diff(F,D). The proposition will follow from the

(1.3.4) LEMMA. $E_x(F)$ is weakly contractible.

Proof: Let $E_o(F)$ denote the subspace of all $f \in E_x(F)$ such that the differential at $0 \in \mathbb{D}$. $D_o f : T_o(\mathbb{D}) \longrightarrow T_p(F)$ is - written in the bases $a = dx$, $b = dy$ and $a' = D_o h(a)$, $b' = D_o h(b)$ - a positive multiple of the identity matrix. Clearly, $E_o(F)$ is a deformation retract of $E_x(F)$. Then we consider the larger space $Imm_o(\mathbb{D},F)$ of all smooth immersions $f : \mathbb{D} \longrightarrow F$ such that $f(0) = P$ and $D_o f$ is a positive multiple of the identity matrix.

The function $\rho(f) = \max\{r \mid f \text{ is injective on } |Z|< r\}$ is continuous on $Imm_o(\mathbb{D},F)$. It is positive, since f is injective in some neighbourhood of 0 . The homotopy $f \longmapsto f_t(z) = f((1-t)\rho(f)z)$, $0 \leq t \leq 1$, retracts $Imm_o(\mathbb{D},F)$ onto the subspace of injective immersions which extend to the closure of $\mathbb{D}$; hence they are embeddings. The differential $D : Imm_o(\mathbb{D},F) \longrightarrow C_o^\infty(\mathbb{D},P(F))$ is a continuous map into the space of smooth maps $\phi : \mathbb{D} \longrightarrow P(F)$, where $P(F)$ is the principal $GL_2^+(\mathbb{R})$-bundle associated to $T(F) \longrightarrow F$, such that $\phi(0)$ is a positive multiple of the identity matrix. D is not surjective, yet a weak homotopy equivalence by Gromov's theory, see [Gromov 1969, 1986], [Haefliger 1969], [Poenaru 1970]. And $C_o^\infty(\mathbb{D},P(F))$ is contractible, because of the normalization condition at 0. ∎

The proposition implies (see [Earle-Schatz 1970, p. 180])

(1.3.5) COROLLARY. $Diff_o(F,\mathbb{D})$ is contractible for $g \geq 2$. ∎

We define the relative mapping class group

(1.3.6) $$\Gamma(F^{o},\partial F^{o}) := \pi_{o}\mathrm{Diff}(F,D) \cong \mathrm{Diff}(F,D)/\mathrm{Diff}_{o}(F,D) .$$

(1.3.7) <u>COROLLARY</u>. $\Gamma(F,x) \cong \Gamma(F,D) = \Gamma(F^{o},\partial F^{o})$. ∎

We summarize the various mapping class groups in a diagram

(1.3.8)

$$\begin{array}{ccccccccc}
\mathrm{Diff}_{o}(F) & \hookleftarrow & \mathrm{Diff}_{o}(F,P) & \hookleftarrow & \mathrm{Diff}_{o}(F,x) & \overset{\simeq}{\hookleftarrow} & \mathrm{Diff}_{o}(F,D) & = & \mathrm{Diff}_{o}(F^{o},\partial F^{o}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Diff}(F) & \hookleftarrow & \mathrm{Diff}(F,P) & \hookleftarrow & \mathrm{Diff}(F,x) & \overset{\simeq}{\hookleftarrow} & \mathrm{Diff}(F,D) & = & \mathrm{Diff}(F^{o},\partial F^{o}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\Gamma(F) & \leftarrow & \Gamma(F,P) & \leftarrow & \Gamma(F,x) & \overset{\cong}{\leftarrow} & \Gamma(F,D) & = & \Gamma(F^{o},\partial F^{o})
\end{array}$$

If the reference surface F is not specified, we write $\Gamma(g)$, $\Gamma^{\cdot}(g)$, $\vec{\Gamma}(g)$ instead of $\Gamma(F)$, $\Gamma(F,P)$, $\Gamma(F,x)$.

Chapter 2

Teichmüller Spaces and Moduli Spaces

The second chapter describes some aspects of Teichmüller theory merely to study the topology of moduli spaces: the fibre bundle description of Teichmüller and moduli spaces, the action of the modular group, and, in particular, the moduli space of "directed Riemann surfaces" as the classifying space of the directed mapping class group.

2.1 Complex structures.

Let V be a 2-dimensional, oriented, real vector space. A complex structure on V is an endomorphism $J : V \longrightarrow V$ such that $J^2(v) = -v$ and $\det(Jv,v) > 0$ for all $v \in V$. The choice of a positively oriented basis of V allows to identify a complex structure with a point of the homogeneous space

(2.1.1) $$C = GL_1(\mathbb{C})\backslash GL_2^+(\mathbb{R}) = \mathbb{R}_+\backslash SO_2(\mathbb{R})\backslash GL_2^+(\mathbb{R})$$

under the correspondence $M \longmapsto J = MJ_oM^{-1}$, $M \in GL_2^+(\mathbb{R})$ and $J_o = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The larger space $SO_2(\mathbb{R})\backslash GL_2^+(\mathbb{R})$ corresponds thereby to positive-definite, symmetric, bilinear forms on V , i.e. to metrics. C is homeomorphic to the unit disc $\mathbb{D}$ via the map

(2.1.2) $$C \longrightarrow \mathbb{D} \ , \quad \begin{bmatrix} a & c \\ b & d \end{bmatrix} \longmapsto \mu \ ,$$

where $\mu = \dfrac{1 + i\tau}{1 - i\tau}$, $\tau = \dfrac{\omega_2}{\omega_1}$ with $\omega_1 = a+ib$, $\omega_2 = c+id$. To $\mu \in \mathbb{D}$ corresponds the conformal class of the Riemannian metric

(2.1.3) $$ds = \lambda|dz + \mu d\bar{z}| \ , \quad \lambda > 0 \ ,$$

on $V = \mathbb{C}$, $z = x+iy$, or in the classical notation

$$ds^2 = Edx^2 + 2Fdxdy + Gdy^2$$

with $E = |1+\mu|^2$, $F = 2\mathrm{Im}(\mu)$, $G = |1-\mu|^2$.

Let F be again as in chapter 1 a connected, closed, compact, oriented and smooth surface. The tangent bundle $T(F)$ is a smooth, oriented vector bundle of rank 2 with structure group $GL_2^+(\mathbb{R})$. Denote the associated smooth bundle with fibre C by $C(F) \longrightarrow F$, and let $S^\infty(X)$ be its space of smooth sections with the C^∞-topology. $S^\infty(F)$ is the space of almost-complex structures on F (compatible with the given orientation); for a surface every almost-complex structure is integrable, thus $S^\infty(F)$ is the space of complex (or conformal) structures on F.

As an example, consider the case when F is a Riemann surface and denote the local parameters by $z_\alpha : Z_\alpha \longrightarrow W_\alpha \subseteq C$ defined on a chart $Z_\alpha \subseteq F$. The derivatives of the transition functions $z_\beta \circ z_\alpha^{-1}$ are in $GL_1(\mathbb{C})$, thus assigning over each chart Z_α the constant section $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ resp. the endomorphism $J_o = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$, yields a well-defined section of $C(F) \longrightarrow F$. One can view this section as a basepoint in $S^\infty(F)$; under the identification (2.1.2) it corresponds to the zero section. Vice versa, given any structure $\mu \in S^\infty(F)$, write $\mu|Z_\alpha$ as a function of the local parameter z_α^{-1}; thus we have a smooth function $\mu_\alpha : W_\alpha \longrightarrow \mathbb{D}$. By the general theory of elliptic differential equations there is a smooth map $q_\alpha : W_\alpha \longrightarrow \tilde{W}_\alpha \subseteq \mathbb{C}$ solving the Beltrami equation

$$\bar{\partial} q_\alpha = \mu_\alpha \cdot \partial q_\alpha .$$

The new parameters $\tilde{z}_\alpha = q_\alpha \circ z_\alpha : Z_\alpha \longrightarrow \tilde{W}_\alpha$ form a new atlas for F. In the complex notation as in (2.1.3) we see that a complex structure $\mu \in S^\infty(F)$ is a smooth differential form of type $(-1,1)$ on F; in other words, it is a smooth section of the complex line bundle $\kappa^{-1} \otimes \bar{\kappa}$ of norm smaller than 1, where κ is the canonical bundle of the complex curve F.

Assume $g \geqq 2$. Then the universal covering space $\tilde{F}$ of F is diffeomorphic to $\mathbb{D}$, and the group of covering transformations, i.e. the fundamental group $\pi_1(F)$, can be realized as a Fuchsian Group $G \leqq SL_2(\mathbb{R})$. Lifting the complex structures from F to $\tilde{F}$, makes $S^\infty(F)$ homeomorphic to the space $S^\infty(\mathbb{D})^G$ of G-invariant complex structures of $\mathbb{D}$. Using (2.1.2), $S^\infty(F) \cong S^\infty(\mathbb{D})^G \subseteq S^\infty(\mathbb{D})$, and $S^\infty(\mathbb{D}) = C^\infty(\mathbb{D},\mathbb{D})$ is the unit ball in the Banach space of smooth, complex functions on $\mathbb{D}$. For a structure μ to be G-invariant is equivalent to

(2.1.4) $\mu = (\mu \circ g)\frac{\overline{g'}}{g'}$ for all $g \in G \subseteq SL_2(\mathbb{R})$,

where g' denotes the (complex) derivative. This shows, that $S^\infty(\mathbb{D})^G$ is a convex subset, and thus we have [Earle-Eells 1969, p.25,26] ,

(2.1.5) Proposition. $S^\infty(F)$ is contractible for $g \geq 2$. ∎

Recall the group Diff(F) of smooth diffeomorphisms of F and its identity component $\mathrm{Diff}_o(F)$. By pulling back sections Diff(F) acts on $S^\infty(F)$,

(2.1.6) $$\gamma.\mu = \gamma^{*-1} \circ \mu \circ \gamma ,$$

$$= \frac{(\mu \circ \gamma^{-1}) + \frac{\bar\partial \gamma}{\overline{\partial \gamma}}}{\frac{\partial \gamma}{\overline{\partial \gamma}} + (\mu \circ \gamma^{-1})\frac{\overline{\bar\partial \gamma}}{\overline{\partial \gamma}}}$$

for $\gamma \in \mathrm{Diff}(F)$, $\mu \in S^\infty(F)$; here γ^* is the map induced by the differential $D\gamma$ on the bundle $C(F) \longrightarrow F$. The following proposition from [Earle-Eells 1969, p.27,28] summarizes what we need to know about this action.

(2.1.7) Proposition.

(i) The action $\mathrm{Diff}(F) \times S^\infty(F) \longrightarrow S^\infty(F)$ is continuous, proper and effective;

(ii) the subgroup $\mathrm{Diff}_o(F)$ acts properly and freely. ∎

The properness means here that the shear map $(\gamma,\mu) \longrightarrow (\mu,\gamma.\mu)$ is a proper map, i.e. the inverse image of compact sets are compact.

2.2 The analytic definition of Teichmüller and moduli spaces.

The Teichmüller space of the smooth surface F has now the analytic definitic

(2.2.1) $\quad \mathfrak{T}(F) := S^{\infty}(F) \,/\, \mathrm{Diff}_o(F)$.

It follows from (2.1.7) and the existence of local sections for the action of $\mathrm{Diff}_o(F)$ [Earle-Eells 1969, p.33] that the orbit projection $S^{\infty}(F) \longrightarrow \mathfrak{T}(F)$ is a universal, principal $\mathrm{Diff}_o(F)$-bundle. The central result about $\mathfrak{T}(F)$ is Teichmüller's theorem.

(2.2.2) Proposition. $\mathfrak{T}(F)$ is homeomorphic to $\mathbb{R}^{6g-g}$ for $g \geq 2$. ∎

The most important consequence for our purposes is the contractibility of $\mathfrak{T}(F)$. By (2.1.5) , this is equivalent to the contractibility of $\mathrm{Diff}_o(F)$. A proof of Teichmüller's theorem in the framework of this chapter is contained in [Fischer-Tromba 1987] . cf. 2.4 .

Recall the groups $\mathrm{Diff}(F,x) \subseteq \mathrm{Diff}_o(F,P) \subseteq \mathrm{Diff}_o(F)$ act also properly, freely and with local sections on $S^{\infty}(F)$. We define the pointed and the directed Teichmüller space

(2.2.3) $\quad \mathfrak{T}(F,P) := S^{\infty}(F)/\mathrm{Diff}_o(F,P)$,

(2.2.4) $\quad \mathfrak{T}(F,x) := S^{\infty}(F)/\mathrm{Diff}_o(F,x)$.

Altogether we have a diagram where all columns are universal, principal, fibre bundles.

(2.2.5)

$$\begin{array}{ccccc} \mathrm{Diff}_o(F) & \supseteq & \mathrm{Diff}_o(F,P) & \supseteq & \mathrm{Diff}_o(F,x) \\ \downarrow & & \downarrow & & \downarrow \\ S^\infty(F) & = & S^\infty(F) & = & S^\infty(F) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{T}(F) & \longleftarrow & \mathcal{T}(F,P) & \longleftarrow & \mathcal{T}(F,x) \end{array}$$

(2.2.6) Proposition. Assume $g \geqq 2$.

(i) The projection $\mathcal{T}(F,P) \longrightarrow \mathcal{T}(F)$ is a trivial fibre bundle with fibre $\mathrm{Diff}_o(F)/\mathrm{Diff}_o(F,P) \cong \tilde{F}$, the universal cover of F;

(ii) the projection $\mathcal{T}(F,x) \longrightarrow \mathcal{T}(F,P)$ is a trivial fibre bundle with fibre $\mathrm{Diff}_o(F,P)/\mathrm{Diff}_o(F,x) \cong \mathbb{R}$, the universal cover of $T_P(F) \cong S^1$;

(iii) the projection $\mathcal{T}(F,x) \longrightarrow \mathcal{T}(F)$ is a trivial fibre bundle with fibre $\mathrm{Diff}_o(F)/\mathrm{Diff}_o(F,x) \cong \vec{T}(F)^{\sim}$, the universal cover of $\vec{T}(F)$.

Proof: It remains to identify the fibres.

(i) Define a map $S : \mathrm{Diff}_o(F) \longrightarrow \tilde{F}$ by $S(\gamma) = (\gamma(P),[w_\gamma])$ where $[w_\gamma]$ is the homotopy class relative endpoints of the track curve $t \longrightarrow w_\gamma(t) := \gamma_t(P)$ $(0 \leq t \leq 1)$, for some homotopy γ_t from $\gamma_o = \mathrm{id}$ to $\gamma_1 = \gamma$. Note that $w_\gamma(0) = P$ for all γ. The homotopy class $[w_\gamma]$ is independent of the choice of a homotopy γ_t, since $\mathrm{Diff}_o(F)$ is simply-connected. S is obviously continuous. It is also surjective, because $\mathrm{Diff}_o(F)$ acts transitively on F and because any curve is the track curve of some homotopy γ_t. Let c be the constant path at P. The fibre over $(P,[c])$ is precisely $\mathrm{Diff}_o(F,P)$. To see that S is a locally trivial bundle, let U be an open disc centered around $P' = \phi(P)$ for some $\phi \in \mathrm{Diff}_o(F)$. There is a continuous family $g_u \in \mathrm{Diff}(F,F-U) \cong \mathrm{Diff}(\bar{U},\partial\bar{U})$, $u \in U$, with the property $g_u(u) = P'$. Let

$\tilde{U}$ be the neighbourhood of $(P,[w_\phi])$ in $\tilde{F}$ which projects onto U under the covering $\tilde{F} \longrightarrow F$. Then

$$(2.2.7) \qquad \begin{array}{ccl} S^{-1}(\tilde{U}) & \longrightarrow & \tilde{U} \times \mathrm{Diff}_o(F,P) \\ \gamma & \longrightarrow & ((\gamma(p),[w_\gamma],\phi^{-1} \circ g_{\gamma(P)} \circ \gamma) \end{array}$$

is a local trivialization.

The proofs for (ii) and (iii) are similar using obvious maps $S' : \mathrm{Diff}_o(F,P) \longrightarrow \vec{T}_P(F)^{\sim}$ and $S'' : \mathrm{Diff}_o(F) \longrightarrow \vec{T}(F)^{\sim}$. They fit into a commutative diagram, which should be compared to (1.2.4) and (1.2.9) .

$$(2.2.8) \qquad \begin{array}{ccccccc} & & \mathrm{Diff}_o(F,P) & \xrightarrow{S'} & \vec{T}_P(F)^{\sim} & \cong & \mathbb{R} \\ & & | & & | & & \\ \mathrm{Diff}_o(F,x) & \longrightarrow & \mathrm{Diff}_o(F) & \xrightarrow{S''} & \vec{T}_P(F)^{\sim} & & \\ | & & \| & & | & & \\ \mathrm{Diff}_o(F,P) & \longrightarrow & \mathrm{Diff}_o(F) & \xrightarrow{S} & \tilde{F} & & \end{array}$$

For another proof see [Nag 1988,p.342] . ■

On the three Teichmüller spaces we still have the action of the full diffeomorphism group. They collapse to actions of the corresponding mapping class groups $\Gamma(F)$, $\Gamma(F,P)$ and $\Gamma(F,x)$. The properness of the Diff(F)-action on $S^\infty(F)$ implies the proper-discontinuity of the actions by the mapping class groups, see [Earle-Eells 1969, p.28] .

(2.2.9) <u>Proposition.</u>

(i) $\Gamma(F)$ <u>acts properly discontinuously on</u> $\mathfrak{T}(F)$;

(ii) $\Gamma(F,P)$ <u>acts properly discontinuously on</u> $\mathfrak{T}(F,P)$;

(iii) $\Gamma(F,x)$ <u>acts properly discontinuously on</u> $\mathfrak{T}(F,x)$. ■

The quotient spaces are the (Riemann) <u>moduli spaces</u>

$$(2.2.10) \qquad \mathfrak{M}(F) := \mathfrak{T}(F)/\Gamma(F) \quad ,$$

$$(2.2.11) \qquad \mathfrak{M}(F,P) := \mathfrak{T}(F,P)/\Gamma(F,P) \quad ,$$

$$(2.2.12) \qquad \mathfrak{M}(F,x) := \mathfrak{T}(F,x)/\Gamma(F,x) \quad .$$

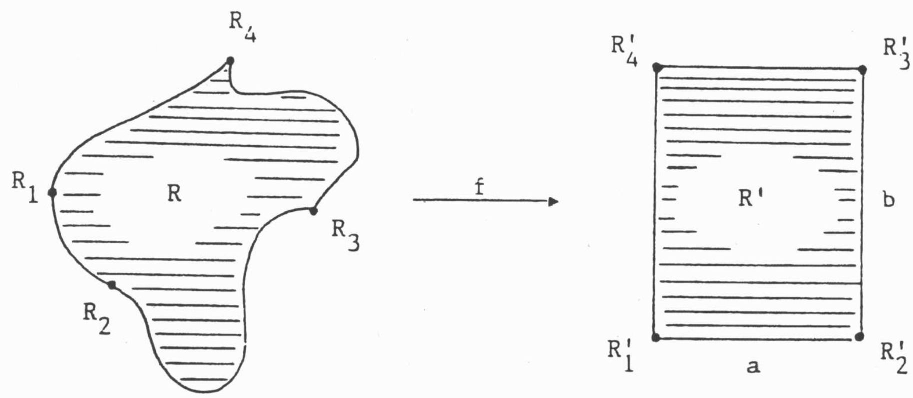
We investigate the actions of the mapping class groups more closely in section 2.4 . So far we know there is a commutative diagram.

$$(2.2.13) \qquad \begin{array}{ccccc} \Gamma(F) & \longleftarrow & \Gamma(F,P) & \longleftarrow & \Gamma(F,x) \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{T}(F) & \longleftarrow & \mathfrak{T}(F,P) & \longleftarrow & \mathfrak{T}(F,x) \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{M}(F) & \longleftarrow & \mathfrak{M}(F,P) & \longleftarrow & \mathfrak{M}(F,x) \end{array}$$

2.3 Quasiconformal mappings.

A quadrilateral $\mathcal{R}$ consists of a simply-connected Jordan domain R in $\mathbb{C}$ with analytic boundary $\partial\bar{R}$ and four distinct points $R_1, R_2, R_3, R_4 \in \partial\bar{R}$ in positive-oriented order. By the Riemann mapping theorem there is a conformal map f from R onto a rectangle R' in $\mathbb{C}$ having one pair of sides paralle to the x-axis and the others parallel to the y-axis, such that $f(R_i) = R_i'$ $(i = 1,\dots,4)$ are the corners of R'. If we require R_1' and R_2' to lie on the x-axis, then f is unique up to a similarity, i.e. a translation and dilatation.

(2.3.1)



The ratio of the width a and the height b of R' is therefore the only conformal invariant of the quadrilateral $\mathcal{R} = (R; R_1, R_2, R_3, R_4)$, and called its <u>modulus</u>,

(2.3.2) $$\mathrm{mod}(\mathcal{R}) = \frac{a}{b} > 0 .$$

Note that $\mathrm{mod}(R; R_2, R_3, R_4, R_1) = \mathrm{mod}(R; R_1, R_2, R_3, R_4)^{-1}$.

Let $q : \Omega \longrightarrow \Omega'$ be a homeomorphism between two regions in $\mathbb{C}$. If $\mathcal{R} = (R; R_1, \dots, R_4)$ is a quadrilateral in Ω, then $q(\mathcal{R}) =$

$= (q(R);q(R_1),\dots,q(R_4))$ is a quadrilateral in Ω' . (N.b. that q is assumed to preserve the orientation.) If there is a number $K > 0$ such that

(2.3.3) $\quad \mathrm{mod}(q(\mathcal{R})) \leqq K\cdot\mathrm{mod}(\mathcal{R})$

for all quadrilaterals $\mathcal{R}$ in Ω , then q is called <u>K-quasiconformal</u>. The smallest such number K is called the <u>(maximal) dilatation</u> and denoted by $K[q]$. If q is K-quasiconformal for some K , it is called quasiconformal.

It is obvious that a conformal map is quasiconformal, that the inverse of a quasiconformal map and the composition of two quasiconformal maps is again quasiconformal. The maximal dilatation has the following properties:

(2.3.4) $\quad K[q] \geqq 1$;

(2.3.5) $\quad K[q] = 1$ if and only if q is conformal ;

(2.3.6) $\quad K[q^{-1}] = K[q]$;

(2.3.7) $\quad K[q_1 \circ q_2] \leqq K[q_1] \cdot K[q_2]$;

(2.3.8) $\quad K[q\circ c_1] = k[q] = [c_2\circ q]$ if c_1, c_2 are conformal.

The assertion (2.3.5) is obvious. One uses (2.3.3) to derive (2.3.4) and (2.3.6) (2.3.7) and (2.3.8) follow immediatedly from the definition. See [Ahlfors 1953] , [Ahlfors 1966, p.22] , [Lehto-Virtanen 1965, p.17] .

Quasiconformality is essentially a local notion [Lehto-Virtanen 1965, p.50] . And $K[q]$ is a conformal invariant by (2.3.8) . This allows to extend the notion to homeomorphisms $q : F \longrightarrow F^*$ between Riemann surfaces. Let $z_\alpha : Z_\alpha \longrightarrow W_\alpha$ resp. $z^*_\beta : Z^*_\beta \longrightarrow W^*_\beta$ be local parameters of F resp. F' . q is called <u>quasiconformal</u>, if the following conditions are satisfied:

(2.3.9) (i) $q_{\beta\alpha} = z^*_\beta \circ q \circ z^{-1}_\alpha$ is a quasiconformal map from $W_{\beta\alpha} = z_\alpha(Z_\alpha \cap q^{-1}(z^*_\beta))$ to $W^*_{\beta\alpha} = z^*_\beta(Z^*_\beta \cap q(z_\alpha))$ for all α, β ;

(ii) there is an upper bound for all $K[q_{\beta\alpha}]$.

The least upper bound of all $K[q_{\beta\alpha}]$ is called the maximal dilatation $K[q]$ of q . We repeat that this number is well-defined. The properties (2.3.4) - (2.3.8) hold now for quasiconformal homeomorphisms between Riemann surfaces.

A quasiconformal homeomorphism need not be differentiable; but it is almost-everywhere differentiable [Lehto-Virtanen 1965, p.172] . For a homeomorphism $q : \Omega \longrightarrow \Omega'$ of class C^1 between two domains in $\mathbb{C}$ one can characterize quasiconformality as follows, [Ahlfors 1953] , [Ahlfors 1966] , [Lehto-Virtanen 1965, p.191] . If $q(z) = q(x,y) = f(x,y) + ig(x,y)$ is considered as a function of x and y with real part f and imaginary part g , on sets

$$\frac{\partial q}{\partial z} = \partial q = \frac{1}{2}\left(\frac{\partial q}{\partial x} - i\frac{\partial q}{\partial y}\right) , \quad \frac{\partial q}{\partial \bar{z}} = \bar{\partial} q = \frac{1}{2}\left(\frac{\partial q}{\partial x} + i\frac{\partial q}{\partial x}\right)$$

with

$$\frac{\partial q}{\partial x} = \frac{\partial f}{\partial x} + i\frac{\partial g}{\partial x} , \quad \frac{\partial q}{\partial y} = \frac{\partial f}{\partial y} + i\frac{\partial y}{\partial y} .$$

The quotient

(2.3.10) $$\mu_q(z) = \frac{\bar{\partial} q(z)}{\partial q(z)} : \Omega \longrightarrow \mathbb{D}$$

is called the <u>complex dilatation</u> of q . We also introduce its norm

(2.3.11) $$k[q] = \|\mu_q\| .$$

Then q is quasiconformal if and only if $k[q] < 1$; its maximal dilatation is

(2.3.12) $$K[q] = \frac{1 + k[q]}{1 - k[q]} .$$

An equation $\bar{\partial} q = \mu \partial q$ is called the Beltrami equation, and therefore μ is called the Beltrami coefficient. If q is a diffeomorphism $F \longrightarrow F'$ between two surfaces, then the complex dilatation μ_q is defined as above by writing q as a function in local parameters; the Beltrami coefficient is then a $(-1,1)$ differential on F. If q is a C_k-diffeomorphism with $k \geq 1$, μ_q will be a C^{k-1}-form and thus continuous. Therefore $\|\mu_q(z)\|$ assumes a maximum if F is compact; it follows that any diffeomorphism between compact surfaces is quasiconformal.

Later we will need some of the following examples and estimates for their dilatation.

The easiest quasiconformal homeomorphism of $\mathbb{C}$ is

(2.3.13) $\quad q(z) = z + k\bar{z}$, for $0 \leq k < 1$, with the constant dilatation $\mu_q = k$.

In general, any linear (or affine) map

(2.3.14)
$$q(z) = M \cdot \begin{pmatrix} x \\ y \end{pmatrix} \qquad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$$
$$= (a+c)x + i(b+d)y$$
$$= \frac{\alpha - i\beta}{z} z + \frac{\alpha + i\beta}{z} \bar{z} \quad , \quad \alpha = a + ic \quad , \quad \beta = b + id$$

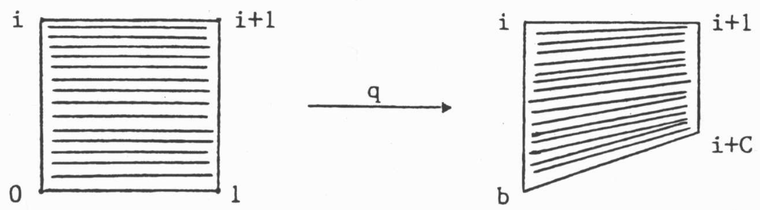
is quasiconformal with constant complex dilatation $\mu_q = \frac{\alpha + i\beta}{\alpha - i\beta} = \frac{1 + i\tau}{1 - i\tau}$, $\tau = \frac{\beta}{\alpha}$. For example, the linear map $q(z) = \dfrac{(\tau_2 - \bar{\tau}_1)z + (\tau_1 - \tau_2)\bar{z}}{\tau_1 - \bar{\tau}_1}$ maps the parallelogram Ω_1 spanned by $0,1$ and τ_1 onto the parallelogram Ω_2 spanned by $0,1$ and τ_2 ; its dilatation is $K[q] = e^t$ where t is the hyperbolic distance between τ_1 and τ_2 . Thus the two parallelograms are K-quasiconformally equivalent if K is the distance between τ_1 and the nearest point equivalent to τ_2 under the action of $SL_2(\mathbf{Z})$ on $\mathbf{H}$. One can show that q is extremal, i.e. ti has the smallest possible maximal dilatations among all homeomorphisms mapping Ω_1 to Ω_2 . See [Ahlfors 1966, p.59] , [Lehto 1988, p.216] . This K is the distance in $\mathbf{H}/SL_2(\mathbf{Z})$.

Particularly usefui mappings will be of the following kind

$$(2.3.15) \qquad q(x,y) = (x, y + Cx(1-y)) \qquad (0 \leq C < 1)$$

$$= (2 + iC)\,z + \frac{C}{2}z^2 + iC\,\bar{z} - \frac{C}{2}\bar{z}^2$$

for $0 \leq x$, $y \leq 1$. q maps a square onto a quadrilateral:



It extends to a neighbourhood of the square and is smooth. For the derivatives we find

$$\bar{\partial}q(z) = C(i-\bar{z}) \quad , \qquad \partial q(z) = 2 + C(i+z) \; .$$

So the complex dilatation is $\mu_q = \dfrac{C(i-\bar{z})}{C(i+z)+2}$. From the inequalities

$$-C \leq \mathrm{Re}(\bar{\partial}q(z)) \leq 0 \quad , \qquad 0 \leq \mathrm{Im}(\bar{\partial}q(z)) \leq 2C \quad ,$$

$$2 \leq \mathrm{Re}(\partial q(z)) \leq 2 + C \quad , \qquad 2C \leq \mathrm{Im}(\partial q(z)) \leq 1 + C$$

we conclude

$$k[q]^2 = \sup_z \frac{|\bar{\partial}q(z)|}{|\partial q(z)|} \leq \frac{C^2 + 4C^2}{4 + 4C^2} = \frac{5C^2}{4(C^2+1)}$$

for the norm square of the complex dilatation. This number is smaller than 1 , since $C < 1$. Therefore q is quasiconformal.

To displace the center of $\mathbb{D}$ to some other point in $\mathbb{D}$ one can use the map

$$(2.3.16) \qquad q(z) = z + (1-z\bar{z})a \quad , \qquad a \in \mathbb{D}$$
$$= a + (1-a\bar{z})z$$

q is a homeomorphism of $\bar{\mathbb{D}}$ such that $q(0) = a$ and $q(z) = z$ for $z \in \partial\bar{\mathbb{D}}$. q is the restriction of a homeomorphism of the entire plane, and except for $z = 0$ it is everywhere differentiable. To prove the conformality we use the concept of circular dilatation, see [Lehto-Virtanen 1965, p.110] .
At $\zeta \in \mathbb{D}$ define

$$H_q(\zeta) = \lim_{r\to 0} \frac{\max\limits_{|\varepsilon|=r} |q(\zeta+\varepsilon) - q(\zeta)|}{\min\limits_{|\varepsilon|=r} |q(\zeta+\varepsilon) - q(\zeta)|} ,$$

which is greater or equal to 1, but possibly ∞. If $H_q(\zeta)$ is finite for all ζ, and $H_q(\zeta) \leq K$ for almost-all ζ, then q is quasiconformal, [Lehto-Virtanen 1965, p.187] . In case q is differentiable at ζ, then

$$H_q(\zeta) = \frac{|\partial q(\partial) + \bar{\partial} q(\zeta)|}{|\partial q(\zeta) - \bar{\partial} q(\zeta)|} = \frac{1 + |\mu_q(\zeta)|}{1 - |\mu_q(\zeta)|} .$$

To make the computation easier we specialize to the case $a \in \mathbb{R}$, $0 < a < \frac{1}{2}$. Then we find

$$q(\zeta+\varepsilon) - q(\zeta) = \varepsilon + a(\zeta\bar{\varepsilon} + \varepsilon\bar{\zeta} + \varepsilon\bar{\varepsilon})$$
$$= \varepsilon + a(2\mathrm{Re}(\zeta\varepsilon) + |\varepsilon|^2) .$$

For $|\varepsilon| = r$ the maximum of the norm satisfies

$$\max \leq r + a(2|\zeta|r + r^2) = r(1 + 2a|\zeta| + ar) ,$$

and for the minimum (for small r)

$$\min \geq r - a(2|\zeta|r - r^2) = r(1 - 2a|\zeta| - ar) .$$

Therefore

$$H_q(\zeta) \leq \limsup_{r\to 0} \frac{1 + 2a|\zeta| + ar}{1 - 2a|\zeta| - ar} = \frac{1 + 2a|\zeta|}{1 - 2a|\zeta|} .$$

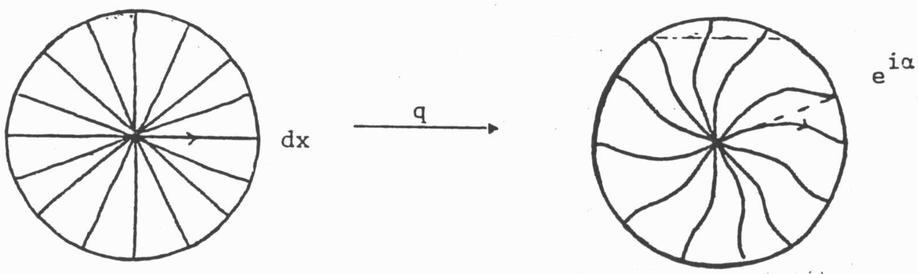
for all $\zeta \in \mathbf{D}$, which gives $K = \frac{1 + 2a}{1 - 2a}$ as an upper bound for the maximal dilatation of q .

We remark that q is not the extremal map solving this mapping problem. The extremal map is the composition of several conformal maps with a single affine mapping of dilatation K_o , cf. [Teichmüller 1944] .

As a last example we consider

(2.3.17) $\quad q(z) = ze^{i\alpha(1-z\bar{z})} , \quad \alpha \in \mathbf{R} , \quad \alpha \geq 0 .$

q is a diffeomorphism of the closed disc $\bar{\mathbf{D}}$, with $q(z) = z$ for $z \in \partial\bar{\mathbf{D}}$ and $q(0) = 0$. The differential at 0 is a rotation about the angel α , $\vec{D}q(\vec{dx}) = e^{i\alpha}\,\vec{dx}$.



To compute the dilatation we find

$$\bar{\partial}q = ze^{i\alpha(1-z\bar{z})}\cdot(-2i\alpha z) = -2i\alpha z^2\cdot e^{i\alpha(1-z\bar{z})} , \quad \text{and}$$

$$\partial q = e^{i\alpha(1-z\bar{z})} + ze^{i\alpha(1-z\bar{z})}(-2i\alpha\bar{z})$$

$$= (1 - 2i\alpha z\bar{z})\cdot e^{i\alpha(1-z\bar{z})}$$

thus $\quad \mu_q = \frac{-2i\alpha z^2}{1 - 2i\alpha z\bar{z}} \quad$ and

$$k[q]^2 = \sup_z \frac{4\alpha^2|z|^4}{|1-2i\alpha|z|^2|^2} = \sup_z \frac{4\alpha^2|z|^4}{4\alpha^2|z|^4+1} = \frac{4\alpha^2}{4\alpha^2+1} .$$

Thus we have a quasiconformal map. There is no extremal map for this mapping problem; by substituting a smooth $\psi : [0,1] \longrightarrow [0,1]$ with $\psi(0) = 1$, $\psi(1) = 0$, $\psi'(0) = \psi'(1) = 0$ such that $q(z) = ze^{i\alpha\psi(1-z\bar{z})}$, one can achieve arbitrarily small dilatations.

After these examples we will see that there are enough morphisms in the category of quasiconformal maps between closed Riemann surfaces of genus g.

(2.3.18) <u>In any homotopy class of homeomorphisms</u> $F_1 \longrightarrow F_2$ <u>there are quasiconformal maps.</u>

See [Lehto 1988, p.181], [Teichmüller 1939, p.27]. But it follows already from the fact that there are diffeomorphisms in each homotopy class. Moreover,

(2.3.19) <u>in each homotopy class there is a unique quasiconformal map with minimal maximal dilatation;</u>

see [Ahlfors 1953, p.16,17], [Lehto 1988, p.231,237], for example. This map is called an extremal map or a Teichmüller map and is, in general, not differentiable if $g \geq 2$.

(2.3.20) <u>If two quasiconformal maps are homotopic as homeomorphisms, they are homotopic as quasiconformal maps.</u>

This follows from the fact, that any deformation class contains real-analytic homeomorphisms, see [Lehto 1988, p.200], [Nag 1988, p.317], [Abikoff 1976, p.31]

Let us define QC(F) to be the group of quasiconformal self-maps of F . QC(F,P) will denote the subgroups of $\gamma \in QC(F)$ satisfying $\gamma(P) = P$, and QC(F,x) the subgroup of such γ which are differentiable at P and satisfy $\vec{D}\gamma(x) = x$. The subscript $_o$ will denote the respective subgroups of those γ being homotopic (relative P , resp. x) to the identity. Given two surfaces F_1 , F_2 , then $QC(F_1;F_2)$, $QC(F_1,P_1;F_2,P_2)$, ... have the obvious meaning. First note, that QC(F) is merely a group and has no topology (for reasons we will explain later). We have

(2.3.21)

$$\begin{array}{ccccc} \mathrm{Diff}(F) & \subseteq & QC(F) & \subseteq & \mathrm{Homeo}(F) \\ \cup\!\mathrm{I} & & \cup\!\mathrm{I} & & \cup\!\mathrm{I} \\ \mathrm{Diff}_o(F) & \subseteq & QC_o(F) & \subseteq & \mathrm{Homeo}_o(F) \end{array} \quad .$$

and similarly for the groups QC(F,P) and QC(F,x) . For the obvious (evaluation) actions of these groups we conclude from the examples (2.3.16) and (2.3.17) the following, (see [Bers 1957/58, p.29], [Nag 1988; p.37] for direct existence proofs.).

(2.3.22) $QC_o(F)$ <u>acts transitively on</u> F ; <u>the subgroup of</u> $\gamma \in QC_o(F)$ <u>being differentiable at</u> P <u>acts transitively on</u> $\vec{T}(F)$; <u>the subgroup of</u> $\gamma \in QC_o(F,P)$ <u>being differentiable at</u> P <u>acts transitive on</u> $T_P(F)$.

As quotients we get the mapping class groups.

(2.3.23) <u>Proposition.</u> (i) $QC(F)/QC_o(F) \cong \Gamma(F)$;

(ii) $QC(F,P)/QC_o(F,P) \cong \Gamma(F,P)$;

(iii) $QC(F,x)/QC_o(F,x) \cong \Gamma(F,x)$.

Proof: (i) is a combination of (2.3.18) and (2.3.20). For (ii) and (iii), note that any mapping class in $\Gamma(F,P)$ and $\Gamma(F,x)$ is represented by an element in Diff(F,P) resp. Diff(F,x) . Assume two quasiconformal maps in QC(F,P) or QC(F,x) are freely homotopic as homeomorphisms. Then they are freely homotopic by quasiconformal maps. There is then a track curve of P , resp. x . Now in the examples (2.3.16) and (2.3.17) the parameter a , resp. α , gives a (continuous) family of quasiconformal maps. Locally, they can be used to re-displace P or x along their track curve. ∎

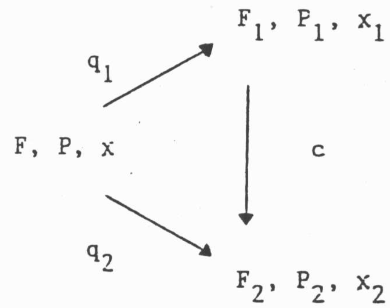
We note that a statement corresponding to (2.3.20) is true for based maps: in any based homotopy class there is a unique based quasiconformal map with minimal maximal dilatation, [Kra 1981]. The corresponding statement for maps preserving a direction does not seem to hold; cf (2.3.17).

2.4 The geometric definition of Teichmüller and moduli spaces.

Let F have a conformal structure, i.e. F is a Riemann surface as in 2.3 .

A deformation of (the conformal structure) of F is a quasiconformal homeomorphism $q : F \longrightarrow F_1$ onto some other Riemann surface. A deformation of the pointed surface (F,P) consists of a surface F_1 , specified point $P_1 \in F_1$ and a quasiconformal homeomorphism $a : F \longrightarrow F_1$ such that $q(P) = P_1$. A deformation of a directed surface (F,x) consists of a surface F_1 , a direction x_1 on F_1 at some point $P_1 \in F_1$, and a quasiconformal homeomorphism $q : F \longrightarrow F_1$ which is differentiable at P and satisfies $q(P) = P_1$ and $\vec{D}q(x) = x_1$. Two deformations q_1 and q_2 of F , resp. (F,P) , resp. (F,x) , are called equivalent, if there is a conformal map $c : F_1 \longrightarrow F_2$, resp. $c : (F_1,P_1) \longrightarrow$ $\longrightarrow (F_2,P_2)$, resp. $c : (F_1,x_1) \longrightarrow (F_2,x_2)$, such that $q_2 = c \circ q_1$.

(2.4.1)



In the pointed case this means $c(P_1) = P_2$, and $\vec{D}c(x_1) = x_2$ in the directed case. An equivalence class is denoted by (q) , resp. (q_i,P_i) , resp. (q_i,x_i) and the sets of equivalence classes by $\mathfrak{D}(F)$, $\mathfrak{D}(F,P)$ and $\mathfrak{D}(F,x)$. There are the following forgetful maps

(2.4.2) $\qquad \mathfrak{D}(F) \longleftarrow \mathfrak{D}(F,P) \longleftarrow \mathfrak{D}(F,x)$.

That the maps are surjective (and the sets are non-empty), is guaranteed by

(2.3.22). The identity $q = \mathrm{id}$ represents a basepoint on these sets.

The <u>Teichmüller metric</u> is defined by

$$(2.4.3) \qquad d((q_1),(q_2)) = \frac{1}{2} \log K[q_2 \circ q_1^{-1}] \quad .$$

Note first that d is well-defined on deformation classes because of (2.3.8). By (2.3.4) $d \geq 0$; and $d((q_1),(q_2)) = 0$ implies $K[q_2 \circ q_1^{-1}] = 1$, thus $q_2 \circ$ is a conformal map c , i.e. an equivalence from q_1 to q_2 . The symmetry follows from (2.3.6) , and the triangle inequality from (2.3.7). The formula defines a metric on each of the sets $\mathcal{D}(F)$, $\mathcal{D}(F,P)$, $\mathcal{D}(F,x)$, now called <u>deformation spaces.</u> In these metrics the maps in (2.4.2) decrease distances.

The groups $QC(F) \supseteq QC(F,P) \supseteq QC(F,x)$ of quasiconformal self-homeomorphisms act on the corresponding deformation spaces by

$$(2.4.4) \qquad \gamma.(q) := (q \circ \gamma) \ , \quad \gamma \in QC(F) \ , \ QC(F,P) \ , \ QC(F,x)$$
$$(q) \in \mathcal{D}(F) \ , \ \mathcal{D}(F,P) \ , \ \mathcal{D}(F,x) \quad .$$

These actions are well-defined and isometric. Let us consider an orbit under the subgroup $QC_o(F)$, $QC_o(F,P)$, resp. $QC_o(F,x)$. Two deformations q_1, q_2 lie in the same orbit if and only if there is a quasiconformal homeomorphism $\gamma : F \longrightarrow F$, homotopic to the identity (relative P , resp. x) such that (q_2) $(q_2) = \gamma.(q_1) = (q_1 \circ \gamma)$; the last statement is equivalent to $c := q_2 \circ \gamma^{-1} \circ q_1^{-1}$ being conformal. In other words, up to a conformal equivalence $c : F_1 \longrightarrow F_2$ the two deformations q_1 , q_2 are homotopic (relative P , resp. x) to each other.

(2.4.5)

$$\begin{array}{ccc} F, P, x & \xrightarrow{\quad q_1 \quad} & F_1, P_1, x_1 \\ {\scriptstyle \mathrm{id} \simeq \gamma} \big\downarrow & & \big\downarrow {\scriptstyle c} \\ F, P, x & \xrightarrow[\quad q_2 \quad]{} & F_2, P_2, x_2 \end{array}$$

Such an orbit is therefore determined by a homotopy class (relative P , resp. of a deformation q . This constitutes a marked Riemann surface (modeled on F where the marking is the homotopy class of q ; see [Nag 1988]).

The Teichmüller spaces of the Riemann surface F could therefore be defined b

(2.4.6) $\mathfrak{T}(F) = \mathfrak{D}(F)/QC_o(F)$,

(2.4.7) $\mathfrak{T}(F,P) = \mathfrak{D}(F,P)/QC_o(F,P)$,

(2.4.8) $\mathfrak{T}(F,x) = \mathfrak{D}(F,x)/QC_o(F,x)$.

It is customary to define the topology not as a quotient topology, but directly with the Teichmüller metric (on the Teichmüller spaces). For two orbits in $\mathfrak{T}(F)$, denoted by $[q_1]$, $[q_2]$, one defines

(2.4.9) $d(\langle q_1\rangle,\langle q_2\rangle) = \frac{1}{2}\log \inf_q K[q]$,

where q runs over the homotopy class of $q_2 \circ q_1^{-1} : F_1 \longrightarrow F_2$ in $QC(F_1;F_2)$. Similarly for $\langle q_1\rangle,\langle q_2\rangle \in \mathfrak{T}(F,P)$ the definition is by the same formula, now q running over the homotopy class of $q_2 \circ q_1^{-1}$ in $QC(F_1,P_1;F_2,P_2)$. In both cases it is immediate from (2.4.3) that d is a pseudo-metric. The existence and uniqueness of Teichmüller maps implies that d is actually a metric: in the free case this is (2.3.19), in the pointed case this is proved in [Kra 1981]. The situation is different in the directed case as we saw at the end of example (2.3.17). Instead of developing a formula as above which takes derivations into account we use an ad-hoc method to remedy the situation. For $\langle q_1\rangle,(\langle q_2\rangle \in \mathfrak{T}(F,x)$ the same formula (2.4.9) is used to define the distance, but we let q run only over all quasiconformal homeomorphisms $F_1 \longrightarrow F_2$ which are homotopic to $q_2 \circ q_1^{-1}$ and agree with $q_2 \circ q_1^{-1}$ in a neighbourhood of $P_1 \in F_1$ (and are therefore differentiable at P_1 and satisfy $q(P_1) = P_2$, and $\vec{D}(x_1) = x_2$.)

The forgetful maps in (2.4.2) induce forgetful maps

(2.4.10) $\mathcal{T}(F) \xleftarrow{t} \mathcal{T}(F,P) \xleftarrow{t'} \mathcal{T}(F,x)$.

Since every Teichmüller class is representable by a smooth deformation, t and t' are surjective.

The following is the classical Teichmüller theorem, see [Ahlfors 1953], [Lehto 1988, p.241].

(2.4.11) Proposition. $\mathcal{T}(F)$ is homeomorphic to $\mathbb{R}^{6g-6}$ for $g \geq 2$. ∎

The basic steps in the proof are as follows.

(1) Any Teichmüller class $\langle q : F \longrightarrow F_o \rangle$ has a unique extremal representative (the Teichmüller map q_T).

(2) If q_T is not conformal, then there are two non-zero quadratic differentials ϕ on F and ϕ_o on F_o , such that q_T has (except at the zeroes of ϕ) the form of an affine map in parameters adopted to ϕ and ϕ_o ; the dilatation is everywhere constant; if ϕ is kept fixed, then ϕ_o is unique up to positive multiples.

(3) The vector space of quadratic differentials on F has dimension $6g-6$ by the Riemann-Roch theorem.

(4) This establishes a correspondence between Teichmüller classes and directions of non-zero quadratic differentials $\psi = q_T^*(\phi_o)$ together with a constant $k \in [0,1]$ with $K[q_t] = \frac{1+k}{1-k}$, which gives a homeomorphism from $\mathcal{T}(F)$ onto the open unit ball in $\mathbb{R}^{6g-6}$.

We need to compare the two definitions of the Teichmüller space, namely

$S^\infty(F)/\mathrm{Diff}_o(F)$ and $\mathcal{D}(F)/QC_o(F)$. A quasiconformal homeomorphism $q : F \longrightarrow$ $\longrightarrow F_o$ (even though it might be non-differentiable at a set of measure zero) always has a complex dilatation μ_q associated; to define this one has to use the generalized derivatives of q , see [Ahlfors 1966], [Lehto 1988]. μ_q is an element in the space of almost-everywhere finite, measurable (-1,1)-differentials on F , denoted by $L^\infty_{-1,1}(F)$. The function $\mathcal{D}(F) \longrightarrow L^\infty_{-1,1}(F)$, $(q) \longmapsto \mu_q$ is well-defined on deformation classes and continuous. Furthermore, it is a homeomorphism and equivariant with respect ot $QC_o(F)$. The space $L^\infty_{-1,1}(F)$ contains $S^\infty(F) = S^\infty_{-1,1}(F)$ as the subspace of smooth Beltrami-differentials. Regarding $\mathrm{Diff}_o(F)$ as a subgroup of $QC_o(F)$, we have an equivariar inclusion $S^\infty(F) \longrightarrow L^\infty_{-1,1}(F)$. Consider the induced map τ between the orbit spaces.

(2.4.12)

$$\begin{array}{ccc} \mathrm{Diff}_o(F) & \longrightarrow & QC_o(F) \\ \downarrow & & \downarrow \\ S^\infty_{-1,1}(F) & \longrightarrow & L^\infty_{-1,1}(F) \\ \downarrow & & \downarrow \\ \mathcal{T}_{an}(F) & \xrightarrow[\tau]{} & \mathcal{T}_{geom}(F) \end{array}$$

Of course, τ is continuous by construction. τ is injective, since $\mathrm{Diff}_o(F)$ is a subgroup. τ is also surjective, since every $QC_o(F)$-orbit in $L^\infty_{-1,1}(F)$ has even a real-analytic representative. Because both sides are homeomorphic to $\mathbb{R}^{6g-6}$, the Brouwer invariance-of-domain theorem implies that τ is a homeomorphism.

(2.4.13) <u>The analytic and geometric Teichmüller space coincide.</u>

To make the picture symmetric, one can also define a smooth deformation space $\mathfrak{C}(F)$. An element is an equivalence class of pairs (f,J) where f is a diffeomorphism $F : F \longrightarrow F_o$ onto some other smooth surface F_o , and J is a complex structure $J \in S^\infty(F_o)$; two pairs (f_1,J_1) and (f_2,J_2) are equivalent, if there is a diffeomorphism $g : F_1 \longrightarrow F_2$ such that $f_2 = g \circ f_1$ and $g^*(J_2) = J_1$. The function $(f,J) \longrightarrow f^*(J)$ induces a bijection $\mathfrak{C}(F) \longrightarrow$ $\longrightarrow S^\infty(F)$, which we declare to be a homeomorphism. With the $\mathrm{Diff}_o(F)$-action $\gamma.(f,J) = (\gamma \circ f, J)$ this homeomorphism is equivariant. The complete picture is

(2.1.14)
$$\begin{array}{ccccccc}
\mathrm{Diff}_o(F) & = & \mathrm{Diff}_o(F) & \longrightarrow & QC_o(F) & = & QC_o(F) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathfrak{C}(F) & \xrightarrow{\cong} & S^\infty_{-1,1}(F) & \longrightarrow & L^\infty_{-1,1}(F) & \xleftarrow{\cong} & \mathfrak{D}(F) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathfrak{T}_{an}(F) & \cong & \mathfrak{T}_{an}(F) & \xrightarrow[\tau]{} & \mathfrak{T}_{geom}(F) & \cong & \mathfrak{T}_{geom}(F)
\end{array}$$

The advantage of the right half is the complex analytic methods, such as the dilatation. The left half has the advantage of the topological groups and the resulting fibre bundles.

To study the other Teichmüller spaces we could argue as above to identify them with their analytically defined counterparts (2.2.6). But we can also stay in the frame work of quasiconformal maps. Consider the projection $t : \mathfrak{T}(F,) \longrightarrow$ $\longrightarrow \mathfrak{T}(F)$, and let $q = (F,P) \longrightarrow (F_o,P_o)$ represent a class in $\langle q \rangle$ in $\mathfrak{T}(F,P)$. Its class $t\langle q \rangle$ in $\mathfrak{T}(F)$ contains a unique extremal map q_T . It may be that $q_T(P) \neq P_o$; but since q is freely homotopic to q_T , such a homotopy gives us a curve w from P to $q_T^{-1}(P_o)$ on F. Its homotopy class relative endpoints is well-defined by $\langle q \rangle$, since the composition of q_T wit a conformal map $c : F_o \longrightarrow F_1$ is the Teichmüller map $(c \circ q)_T$ for the class $\langle c \circ q \rangle$. Thus we have a map $\mathfrak{T}(F,P) \longrightarrow \tilde{F}$. For the continuity we appeal to results of [Kra 1981]. For $t^{\cdot}$ and $t' = t \circ t^{\cdot}$ similar maps onto $\tilde{T}(F)^{\sim} \cong R$ and onto $\vec{T}(F)^{\sim}$ can be constructed. They exhibit $\mathfrak{T}(F,P)$ and $\mathfrak{T}(F,x)$ as products $(g \geq 2)$:

(2.4.15) $$\mathfrak{T}(F,P) \cong \mathfrak{T}(F) \times \tilde{F} \cong R^{6g-4},$$

(2.4.16) $$\mathfrak{T}(F,x) \cong \mathfrak{T}(F,P) \times \widetilde{S^1} \cong R^{6g-3}.$$

Compare this to [Bers 1970]. The group $QC(F)$ acts on $\mathfrak{T}(F)$ by

(2.4.17) $$\gamma.\langle q \rangle := \langle q \circ \gamma \rangle \qquad \gamma \in QC(F), \ \langle q \rangle \in \mathfrak{T}(F).$$

The same formula defines actions of $QC(F,P)$ on $\mathfrak{T}(F,P)$, and of $QC(F,x)$ on $\mathfrak{T}(F,x)$. It is easy to see, that $M_\gamma : \langle q \rangle \longrightarrow \gamma.\langle q \rangle$ is an isometry. An isometry of $\mathfrak{T}(F)$ induced in this way by a quasiconformal self-map of F is usually called allowable, we call it a <u>modular isometry</u>. They constitute the <u>modular group</u> $Mod(F)$ of (the Teichmüller space of) F. In the extensions

$$\begin{array}{lccccccccc}
(2.4.18) & 1 & \longrightarrow & QC^o(F) & \longrightarrow & QC(F) & \longrightarrow & Mod(F) & \longrightarrow & 1 \\
 & & & \uparrow & & \uparrow & & \uparrow & & \\
(2.4.19) & 1 & \longrightarrow & QC^o(F,P) & \longrightarrow & QC(F,P) & \longrightarrow & Mod(F,P) & \longrightarrow & 1 \\
 & & & \uparrow & & \uparrow & & \uparrow & & \\
(2.4.20) & 1 & \longrightarrow & QC^o(F,x) & \longrightarrow & QC(F,x) & \longrightarrow & Mod(F,x) & \longrightarrow & 1
\end{array}$$

the kernels $QC^o(F)$, $QC^o(F,P)$ and $QC^o(F,x)$ obviously contain $QC_o(F)$, $QC_o(F,P)$ resp. $QC_o(F,x)$, since any γ homotopic to the identity acts trivially on the Teichmüller space, i.e. $M_\gamma = id$. With the exceptions of a few cases one always finds $QC_o(F) = QC^o(F)$, and therefore the modular groups are the mapping class groups.

Among the closed surfaces there is only one exception, namely $g = 2$. Any Riemann surface F_2 of genus 2 is hyperelliptic [Farkas-Kra 1980, pp.94,101-103] and allows a conformal involution with 6 fixed points, realizable as the deck transformation of a double covering $F_2 \longrightarrow S^2$ branched over 6 points In this case $\Gamma(2)/\mathbb{Z}_2 \cong Mod(2)$.

This explains why the notion of mapping class group and modular group are treated synonymously in the literature. We will not persue the modular groups. For completeness we mention that $Mod(F)$ comprises in fact all isometries of $\mathfrak{T}(F)$ for $g \geq 3$, [Royden 1971].

For the remaining action of the mapping class groups we quote from [Gardiner 1987, p.149], that $\Gamma(F)$ acts properly discontinuously on $\mathfrak{T}(F)$. The other two actions are compatible via the forgetful maps; for $\gamma \in QC(F,x)$ we have commutative diagrams

$$\begin{array}{ccccc}
\mathfrak{T}(F) & \xleftarrow{t} & \mathfrak{T}(F,P) & \xleftarrow{t'} & \mathfrak{T}(F,x) \\
\downarrow M_\gamma & & \downarrow M_\gamma & & \downarrow M_\gamma \\
\mathfrak{T}(F) & \xleftarrow[t]{} & \mathfrak{T}(F,P) & \xleftarrow[t']{} & \mathfrak{T}(F,x)
\end{array}$$

Furthermore, if $[\gamma] \in \Gamma(F,P)$ is contained in the subgroup $\pi_1(F)$, then M_γ restricts to the corresponding deck transformation on the fibre $\tilde{F}$ of t ; and if $[\gamma] \in \Gamma(F,x)$ is contained in the infinite cyclic subgroup of twists around P , then M_γ restricts to a translation on the fibre $\mathbb{R} \cong T_P(F)^{\sim}$ of $t^{\cdot}$. We have

(2.4.21) Proposition. The actions of $\Gamma(F)$, $\Gamma(F,P)$ and $\Gamma(F,x)$ on the corresponding Teichmüller space $\mathfrak{T}(F)$, $\mathfrak{T}(F,P)$ and $\mathfrak{T}(F,x)$ are properly discontinuous. ▪

Dividing the Teichmüller spaces by the remaining action of the mapping class groups (equivalently of the modular groups) gives us the moduli spaces.

(2.4.22) $\mathfrak{M}(F) := \mathfrak{T}(F)/\Gamma(F)$,

(2.4.23) $\mathfrak{M}(F,P) := \mathfrak{T}(F,P)/\Gamma(F,P)$,

(2.4.24) $\mathfrak{M}(F,x) := \mathfrak{T}(F,x)/\Gamma(F,x)$.

It is clear from our discussion in (2.4.13) that the analytic definition and this geometric definition give homeomorphic moduli spaces.

Two points $\langle q_1\rangle, \langle q_2\rangle \in \mathfrak{T}(F)$ are in the same $\Gamma(F)$-orbit if and only if there is a quasiconformal $\gamma : F \longrightarrow F$ and a conformal map $c : F_1 \longrightarrow F_2$ such that $c \circ q_1 \circ \gamma = q_2$.

(2.4.25)
$$\begin{array}{ccc} F & \xrightarrow{q_1} & F_1 \\ \gamma \uparrow & & \downarrow c \\ F & \xrightarrow[q_2]{} & F_2 \end{array}$$

But trivially, $\gamma := q_1^{-1} \circ c^{-1} \circ q_2$ will do, if only c exists. Thus, the markings have become irrelevant; $\mathfrak{M}(F)$ is the space of conformal equivalence classes $[F_1]$ of surfaces F_1 homeomorphic to F . The metric becomes

(2.4.26) $$\tau([F_1],[F_2]) = \frac{1}{2}\inf_q \log K[q] \quad ,$$

where q runs over $QC(F_1,F_2)$.

Correspondingly, $\mathfrak{M}(F,P)$ resp. $\mathfrak{M}(F,x)$ is the space of conformal equivalence classes $[F_o,P_o]$ resp. $[F_o,x_o]$ of Riemann surfaces F_o with a specified point $P_o \in F_o$ resp. with a specified direction x_o at some point $P_o \in F_o$; two such surfaces are conformally equivalent if there is a conformal map $c : F_1 \longrightarrow F_2$ such that $c(P_1) = P_2$ resp. $\vec{D}c(x_1) = x_2$; the Teichmülle distance is given by the formula (2.4.26) where q runs over $QC(F_1,P_1;F_2,P_2)$ resp. over all quasiconformal homeomorphisms $q : F_1 \longrightarrow F_2$ which are conformal in a neighbourhood of P_1 .

Next we want to determine the fibres of the orbit projections from the Teichmüller spaces to the moduli spaces.

(2.4.27) Proposition.

(i) The isotropy subgroup of $\langle q\rangle = \langle q : F \longrightarrow f_o\rangle \in \mathfrak{T}(F)$ is $Aut(F_o)$, the group of conformal self-maps of F_o;

(ii) the isotropy subgroup of $\langle q\rangle = \langle q : (F,P) \longrightarrow (F_o,P_o)\rangle \in \mathfrak{T}(F,P)$ is $Aut(F_o,P_o)$, the group of pointed conformal self-maps $c : F_o \longrightarrow F_o$ such that $c(P_o) = P_o$;

(iii) the isotropy subgroup of $\langle q\rangle = \langle q : (F,x) \longrightarrow (F_o,x_o)\rangle \in \mathfrak{T}(F,x)$ is $Aut(F_o,x_o)$, the group of directed conformal self-maps $c : F_o \longrightarrow F_o$ such that $\vec{D}c(x_o) = x_o$, which is trivial. ∎

The proof is straightforward from the definition.

$Aut(F_o,P_o)$ is at most cyclic and finite; $Aut(F_o)$ is finite (with at most $84(g-1)$ elements). In cases (i) and (ii) the isotropy group varies with $[F_o]$ resp. $[F_o,P_o]$. Thus $\mathfrak{T}(F) \longrightarrow \mathfrak{M}(F)$ and $\mathfrak{T}(F,P) \longrightarrow \mathfrak{M}(F,P)$ are not (unbranched) coverings. In case (iii) we have

(2.4.28) Proposition. The action

$$\Gamma(F,x) \times \mathfrak{T}(F,x) \longrightarrow \mathfrak{T}(F,x)$$

is free, and the orbit map

$$\mathfrak{T}(F,x) \longrightarrow \mathfrak{M}(F,x)$$

is a universal, principal $\Gamma(F,x)$-bundle.

Proof: The assertions follow from triviality of the isotropy groups, the universality from the contractibility of $\mathfrak{T}(F,x)$, (2.4.16). ∎

(2.4.29) Corollary. $\mathfrak{M}(F,x) \simeq B\Gamma(F,x)$.

We summarize the situation in a diagram.

(2.4.30)
$$\begin{array}{ccccc} \Gamma(F) & \longleftarrow & \Gamma(F,P) & \longleftarrow & \Gamma(F,x) \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{T}(F) & \longleftarrow & \mathfrak{T}(F,P) & \longleftarrow & \mathfrak{T}(F,x) \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{M}(F) & \longleftarrow & \mathfrak{M}(F,P) & \longleftarrow & \mathfrak{M}(F,x) \end{array}$$

The first row are group extensions, the middle row are trivial fibre bundles; the last column is a covering.

If $f : F \longrightarrow F'$ is quasiconformal (and $F(P) = P'$, $\vec{D}f(x) = x'$) it induces isometrics via $f^{*}[q] = [q \circ f]$,

(2.4.31)
$$\begin{array}{ccccc} \mathfrak{M}(F) & \longleftarrow & \mathfrak{M}(F,P) & \longleftarrow & \mathfrak{M}(F,x) \\ \uparrow f^{*} & & \uparrow f^{*} & & \uparrow f^{*} \\ \mathfrak{M}(F') & \longleftarrow & \mathfrak{M}(F',P') & \longleftarrow & \mathfrak{M}(F,x') \end{array} .$$

If the reference surface F (represented by $[F]$ in the moduli spaces) is irrelevant, we write $\mathfrak{M}(g)$, $\mathfrak{M}^{\cdot}(g)$ and $\vec{\mathfrak{M}}(g)$.