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Hand in at the lecture on Tuesday, 2018-04-17

Exercise 1. Let $p \in (0,1)$ and $p' = p/(p-1) < 0$. Define $\|f\|_{p'} = \||f|^{-1}\|_{|p'|}^{-1}$.

(a) Suppose that $f, g > 0$ almost everywhere, $f \in L^p$ and $g^{-1} \in L^{|p'|}$. Show that $\|fg\|_1 \geq \|f\|_p \|g\|_{p'}$.

(b) Find $f, g \in L^p(\mathbb{R})$ such that $\|f+g\|_p > \|f\|_p + \|g\|_p$.

(c) Let $n \in \mathbb{N}$. Show that

$$\left\| \sum_{i=1}^n f_i \right\|_p \leq n^{(1-p)/p} \sum_{i=1}^n \|f_i\|_p.$$

Exercise 2. Let $n \geq 2$.

(a) (Loomis–Whitney inequality). Let $f_j : \mathbb{R}^{n-1} \to \mathbb{R}$ be measurable functions and let $\pi_j(x_1, \dots, x_n) = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$. Prove that

$$\int_{\mathbb{R}^n} \prod_{j=1}^n |f_j \circ \pi_j|^{1/(n-1)} \, dx \leq \prod_{j=1}^n \left(\int_{\mathbb{R}^{n-1}} |f_j| \, dx \right)^{1/(n-1)}.$$

(b) (Sobolev's inequality). Assume that $f \in L^1$ is continuously differentiable and $\partial_j f \in L^1$ for all $j \in \{1, \dots, n\}$. Using the Loomis–Whitney inequality, prove

$$\|f\|_{n/(n-1)} \leq \sum_{j=1}^n \|\partial_j f\|_1.$$